Recursive Algorithms for Calculating the Scattering from N Strips or Patches

LEVENT GUREL, STUDENT MEMBER, IEEE, AND WENG CHO CHEW, SENIOR MEMBER, IEEE

Abstract—**With the definition of recursive relations for the reflection operator for N strips or patches, two recursive algorithms are developed, which are easily programmable, to calculate the scattering by N strips or patches. One algorithm is for arbitrary excitation while the other is for a fixed excitation. The recursive algorithms require the inversion of small matrices at each stage, and hence, are suitable for programming on smaller computers. Also, if the N strips or patches are identical and equally spaced, symmetry can be exploited to speed up the algorithms. A program has been developed by the authors to calculate the scattering by N strips, and the result is shown to converge to the scattering by a large strip when the N strips are touching each other.**

I. Introduction

IN THIS PAPER, we present two efficient recursive algorithms to solve the electromagnetic problems involving perfectly conducting strips or patches placed arbitrarily in a homogeneous medium. These techniques are developed using the concepts of reflection operators and double dot products that we introduced recently [1]–[3]. The double dot product was defined so that complicated expressions could be written in a very compact form, which allows us to handle more difficult, more complex, and larger problems. As will be shown, the use of reflection operators better elucidates the physics of the problems. With this understanding, an efficient recursive technique can be developed, which is easily programmable. The reflection operator due to N strips or patches in a homogeneous medium can be computed in a recursive manner in terms of the reflection operator of the $N-1$ strips or patches. The idea is also similar to a recursive algorithm for studying wave propagation in heterogeneous media [4]. If the strips or patches are in a layered medium instead of a homogeneous one, the problem can be solved with a similar method given in [1]–[3]. Also, such a problem can be broken into smaller, homogeneous-medium problems in order to use the methods of this paper.

There is much interest in solving the electromagnetic problems involving strips or patches in homogeneous or layered media, be it guidance, resonance or scattering problems. Examples are found in complicated microwave integrated circuits (MICs) and in high-speed computer circuits where there are several transmission lines and/or patches used in the same device. Usually, the metallic strips and patches are not on the same plane in these geometries, which makes the problem harder than those involving only coplanar geometries. The method of assigning unknown electric and magnetic potentials in the regions separating the sources and then solving for these unknowns symbolically in the formulation stage becomes laboriously painful as the number of sources at different planes increases [5], [6]. More efficient approaches in solving this problem were given by the authors [1], [2], Itoh [7], and Krowne [8], and will be given in this paper.

Another problem for which the results of this paper may be useful is that of a finite sized frequency selective surface (FSS). Such a surface is made up of a finite number of identical strips (slots) or patches (apertures) placed periodically on the same plane [9]–[12]. A similar problem is the finite phased array of microstrip patches [13].

The strip problems and the patch problems are very similar. The only difference is that strip problems involve one-dimensional spectral integrals whereas patch problems require two-dimensional spectral integrals. Since we are using double dot products in our formulation, this difference is invisible until the computational stage. Therefore, this paper presents a unified approach to both types of problems. In the rest of the paper, we will only use "strips" to imply "strips and patches."

In the recursive algorithms developed here, we find the reflection operator of the N strips given the knowledge of the reflection operator for $N-1$ strips. Hence, with the addition of the Nth strip to the $N-1$ strips, only the part of the solution resulting from the presence of the Nth strip needs to be considered. For instance, if the reflection operator of (and thus the scattering from) a single strip is known, it can be used to find the reflection operator of two strips without having to solve the problem completely from the beginning. This recursive scheme, unlike the conjugate gradient method, corresponds to a physical problem at each stage of the recursion. Our recursive algorithm is more general than the "add-on" scheme [14], [15]. It is valid for arbitrary excitation, and arbitrary number of unknowns (basis functions) can be added at each stage of recursion. In this paper, our formulation is general enough to handle any planar current distribution as the incident source although the presented results will be due to plane wave incidence. In the case of plane wave incidence, we do not have to approximate the incident field with an equivalent fringe current as in the "add-on" technique. The incident plane wave is not restricted to be transverse electric (TE) or transverse magnetic (TM), but can be polarized arbitrarily.

Manuscript received October 20, 1988; revised April 6, 1989. This work was supported by National Science Foundation Grant NSF ECS 85-52891 and by TRW.

The authors are with the Electromagnetics Laboratory, Department of Electrical and Computer Engineering, University of Illinois, Urbana, IL 61801.

IEEE Log Number 9034331.

0018-926X/90/0400-0507$01.00

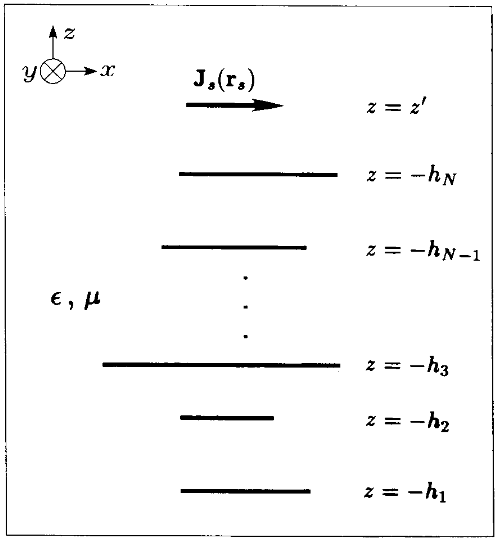


Fig. 1. N strips placed arbitrarily in a homogeneous medium.

II. A Recursive Relation for Reflection Operators

We showed in a previous paper [1] that the process of scattering from strips (and patches) in a homogeneous medium can be characterized by reflection and transmission operators. Fig. 1 shows the geometry of N strips which are arbitrarily placed in a homogeneous medium. If we can find the reflection operator $\overline{\mathcal{R}}_N$ at the reference plane $z = -h_N$, the total transverse electric field for $z > -h_N$ can simply be written as

$$\mathbf{E}_s(\mathbf{r}) = \bar{\mathbf{F}}(\mathbf{r}_s) : (e^{i\kappa_z|z-z'|} + e^{i\kappa_z(z+h_N)} : \overline{\mathcal{R}}_N : e^{i\kappa_z(z'+h_N)}) : \overline{\mathcal{G}} : \tilde{\mathbf{J}}_s. \quad (1)$$

However, $\overline{\mathcal{R}}_N$ can be found from a straightforward generalization of the recursive relation we derived in [1] under the name of the reflection operator due to a strip over a layered medium (or due to two strips). The recursive relation for $\overline{\mathcal{R}}_N$ reads [1, eq. (52)]

$$\begin{aligned}\overline{\mathcal{R}}_N &= e^{i\kappa_z(h_{N-1}-h_N)} : \overline{\mathcal{R}}_{N-1} : e^{i\kappa_z(h_{N-1}-h_N)} \\ &\quad - (\bar{\mathcal{I}} + e^{i\kappa_z(h_{N-1}-h_N)} : \overline{\mathcal{R}}_{N-1} : e^{i\kappa_z(h_{N-1}-h_N)}) \\ &\quad : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_N^t \cdot \bar{\mathbf{\Gamma}}_N{}^{-1} \cdot \tilde{\mathbf{f}}_{Nt} : (\bar{\mathcal{I}} + e^{i\kappa_z(h_{N-1}-h_N)} \\ &\quad : \overline{\mathcal{R}}_{N-1} : e^{i\kappa_z(h_{N-1}-h_N)})\end{aligned} \quad (2)$$

where

$$\bar{\mathbf{\Gamma}}_N = \tilde{\mathbf{f}}_{Nt} : (\bar{\mathcal{I}} + e^{i\kappa_z(h_{N-1}-h_N)} : \overline{\mathcal{R}}_{N-1} : e^{i\kappa_z(h_{N-1}-h_N)}) : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_N^t, \quad (3)$$

and $\overline{\mathcal{R}}_0 = 0$ for the initial value of the recursion.

The simplest example that can demonstrate the use of this recursive relation is the derivation of the reflection operator $\overline{\mathcal{R}}_2$ due to two strips at $z = -h_2$ and $z = -h_1$ where $-h_2 > -h_1$. This example will also serve to establish our notation. Using $N = 2$ in (2), $\overline{\mathcal{R}}_2$ is simply given by

$$\begin{aligned}\overline{\mathcal{R}}_2 &= e^{i\kappa_z(h_1-h_2)} : \overline{\mathcal{R}}_1 : e^{i\kappa_z(h_1-h_2)} \\ &\quad - (\bar{\mathcal{I}} + e^{i\kappa_z(h_1-h_2)} : \overline{\mathcal{R}}_1 : e^{i\kappa_z(h_1-h_2)}) \\ &\quad : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_2^t \cdot \bar{\mathbf{\Gamma}}_2{}^{-1} \cdot \tilde{\mathbf{f}}_{2t} : (\bar{\mathcal{I}} + e^{i\kappa_z(h_1-h_2)} \\ &\quad : \overline{\mathcal{R}}_1 : e^{i\kappa_z(h_1-h_2)}).\end{aligned} \quad (4)$$

In [1], $\overline{\mathcal{R}}_1$ was derived to be

$$\overline{\mathcal{R}}_1 = -\overline{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \cdot \bar{\mathbf{\Gamma}}_1{}^{-1} \cdot \tilde{\mathbf{f}}_{1t}, \quad (5)$$

where

$$\bar{\mathbf{\Gamma}}_1 = \tilde{\mathbf{f}}_{1t} : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_1^t, \quad (6)$$

and hence, is the matrix representation of the operator $\overline{\mathcal{G}}$ in the space spanned by the expansion function $\tilde{\mathbf{f}}_{1t}$ and the testing function $\tilde{\mathbf{f}}_1^t$. $\bar{\mathbf{\Gamma}}_1$ is a reduced Green's operator that maps the space spanned by the expansion function for the current on strip 1 to the space spanned by the testing function for the tangential electric field on strip 1.

Note that the above algorithm has computational time that grows as N (i.e., linear in N) where N is the number of strips. However, the recursive relations (2) and (3) are difficult to compute because they involve double dot products that represent infinite integrals. Furthermore, $\overline{\mathcal{R}}_N$ is an infinite dimensional, continuously indexed operator, which is very difficult to store. We are still investigating this linear in N algorithm. In the following, we shall derive two other recursive relations for $\overline{\mathcal{R}}_N$ that are more easily programmable and computable. The two recursion schemes are equivalent except that one of them is for arbitrary excitation whereas the other is valid only for fixed excitation.

A. Recursive Relation for Arbitrary Excitation

Substituting the expression for $\overline{\mathcal{R}}_1$ in (4), multiplying out all the factors and combining the two similar terms into one, we obtain

$$\begin{aligned}\overline{\mathcal{R}}_2 &= -e^{i\kappa_z(h_1-h_2)} : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \\ &\quad \cdot (\bar{\mathbf{\Gamma}}_1{}^{-1} + \bar{\mathbf{\Gamma}}_1{}^{-1} \cdot \bar{\mathbf{S}}_{12} \cdot \bar{\mathbf{\Gamma}}_2{}^{-1} \cdot \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{\Gamma}}_1{}^{-1}) \cdot \tilde{\mathbf{f}}_{1t} : e^{i\kappa_z(h_1-h_2)} \\ &\quad - e^{i\kappa_z(h_1-h_2)} : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \cdot \bar{\mathbf{\Gamma}}_1{}^{-1} \cdot \bar{\mathbf{S}}_{12} \cdot \bar{\mathbf{\Gamma}}_2{}^{-1} \cdot \tilde{\mathbf{f}}_{2t} \\ &\quad - \overline{\mathcal{G}} : \tilde{\mathbf{f}}_2^t \cdot \bar{\mathbf{\Gamma}}_2{}^{-1} \cdot \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{\Gamma}}_1{}^{-1} \cdot \tilde{\mathbf{f}}_{1t} : e^{i\kappa_z(h_1-h_2)} \\ &\quad - \overline{\mathcal{G}} : \tilde{\mathbf{f}}_2^t \cdot \bar{\mathbf{\Gamma}}_2{}^{-1} \cdot \tilde{\mathbf{f}}_{2t}\end{aligned} \quad (7)$$

where we have the definitions

$$\bar{\mathbf{S}}_{ij} = -\tilde{\mathbf{f}}_{it} : e^{i\kappa_z(h_j-h_i)} : \overline{\mathcal{G}} : \tilde{\mathbf{f}}_j^t, \qquad i, j = 1, 2. \quad (8)$$

With the proper choice of the basis and testing functions [2], we have the relation $\bar{\mathbf{S}}_{12} = \bar{\mathbf{S}}_{21}^t$. The $\bar{\mathbf{S}}_{ij}$ matrix is the matrix representation of the Green's operator, $e^{i\kappa_z(h_j-h_i)} : \overline{\mathcal{G}}$, that maps the currents on the ith strip to the field on the jth strip. We can think of $\bar{\mathbf{S}}_{ij}$ as a propagator matrix from strip j to strip i. Similarly, the $\bar{\mathbf{S}}_{ii}$ matrix represents the self-action of the current on the ith strip. From (6) we see that $\bar{\mathbf{\Gamma}}_1 = -\bar{\mathbf{S}}_{11}$.

From the definition of the $\bar{\mathbf{\Gamma}}_N$ in (3), we observe that

$$\begin{aligned}\bar{\mathbf{\Gamma}}_2 &= \tilde{\mathbf{f}}_{2t} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_2^t + \tilde{\mathbf{f}}_{2t} : e^{i\kappa_z(h_1-h_2)} : \bar{\mathcal{R}}_1 : e^{i\kappa_z(h_1-h_2)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_2^t \\ &= -\bar{\mathbf{S}}_{22} - \tilde{\mathbf{f}}_{2t} : e^{i\kappa_z(h_1-h_2)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \cdot \bar{\mathbf{\Gamma}}_1^{-1} \cdot \tilde{\mathbf{f}}_{1t} \\ &\quad : e^{i\kappa_z(h_1-h_2)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_2^t \\ &= -\bar{\mathbf{S}}_{22} - \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{\Gamma}}_1^{-1} \cdot \bar{\mathbf{S}}_{12}. \end{aligned} \tag{9}$$

By using (9) in the first and the last terms of (7), we can show that the expression is symmetrical about strips 1 and 2, i.e., the first and the last terms are of the same structure. We can rewrite (7) in a more suggestive form as

$$\begin{aligned}\bar{\mathcal{R}}_2 &= -e^{i\kappa_z(h_1-h_2)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \cdot \bar{\mathbf{I}}_{11(2)} \cdot \tilde{\mathbf{f}}_{1t} : e^{i\kappa_z(h_1-h_2)} \\ &\quad - e^{i\kappa_z(h_1-h_2)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \cdot \bar{\mathbf{I}}_{12(2)} \cdot \tilde{\mathbf{f}}_{2t} \\ &\quad - \bar{\mathcal{G}} : \tilde{\mathbf{f}}_2^t \cdot \bar{\mathbf{I}}_{21(2)} \cdot \tilde{\mathbf{f}}_{1t} : e^{i\kappa_z(h_1-h_2)} \\ &\quad - \bar{\mathcal{G}} : \tilde{\mathbf{f}}_2^t : \bar{\mathbf{I}}_{22(2)} \cdot \tilde{\mathbf{f}}_{2t} \\ &= -\sum_{i=1}^{2}\sum_{j=1}^{2} e^{i\kappa_z(h_i-h_2)} \\ &\quad : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_i^t \cdot \bar{\mathbf{I}}_{ij(2)} \cdot \tilde{\mathbf{f}}_{jt} : e^{i\kappa_z(h_j-h_2)} \end{aligned} \tag{10}$$

where we have defined new $\bar{\mathbf{I}}_{ij(2)}$ matrices, which can be thought of as interaction matrices. Similarly, we can rewrite $\bar{\mathcal{R}}_1$ in (5) for the one-strip problem as

$$\bar{\mathcal{R}}_1 = -\bar{\mathcal{G}} : \tilde{\mathbf{f}}_1^t \cdot \bar{\mathbf{I}}_{11(1)} \cdot \tilde{\mathbf{f}}_{1t}, \tag{11}$$

where $\bar{\mathbf{I}}_{11(1)} = \bar{\mathbf{\Gamma}}_1^{-1}$. These matrices represent the interaction of the ith strip with the jth strip in the two-strip problem. Comparing (10) to (7), the $\bar{\mathbf{I}}_{ij(2)}$ matrices are found to be

$$\begin{aligned}\bar{\mathbf{I}}_{11(2)} &= \bar{\mathbf{\Gamma}}_1^{-1} + \bar{\mathbf{\Gamma}}_1^{-1} \cdot \bar{\mathbf{S}}_{12} \cdot \bar{\mathbf{\Gamma}}_2^{-1} \cdot \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{\Gamma}}_1^{-1} \\ &= \bar{\mathbf{I}}_{11(1)} + \bar{\mathbf{I}}_{11(1)} \cdot \bar{\mathbf{S}}_{12} \cdot \bar{\mathbf{\Gamma}}_2^{-1} \cdot \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{I}}_{11(1)} \end{aligned} \tag{12a}$$

$$\begin{aligned}\bar{\mathbf{I}}_{12(2)} &= \bar{\mathbf{\Gamma}}_1^{-1} \cdot \bar{\mathbf{S}}_{12} \cdot \bar{\mathbf{\Gamma}}_2^{-1} \\ &= \bar{\mathbf{I}}_{11(1)} \cdot \bar{\mathbf{S}}_{12} \cdot \bar{\mathbf{\Gamma}}_2^{-1} \end{aligned} \tag{12b}$$

$$\begin{aligned}\bar{\mathbf{I}}_{21(2)} &= \bar{\mathbf{\Gamma}}_2^{-1} \cdot \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{\Gamma}}_1^{-1} \\ &= \bar{\mathbf{\Gamma}}_2^{-1} \cdot \bar{\mathbf{S}}_{21} \cdot \bar{\mathbf{I}}_{11(1)} \end{aligned} \tag{12c}$$

$$\bar{\mathbf{I}}_{22(2)} = \bar{\mathbf{\Gamma}}_2^{-1} \tag{12d}$$

where $\bar{\mathbf{I}}_{11(1)}$ represents the self-interaction of the only strip in the single-strip problem. Again with the proper choice of the basis functions [2], it can be shown that $\bar{\mathbf{I}}_{12(2)} = \bar{\mathbf{I}}_{21(2)}^t$.

In the above, we note that $\bar{\mathcal{R}}_2$ has an intrinsic form given by (10), each term of which can be associated with a scattering picture in Fig. 2. Hence, this form can be generalized to N strips. The reflection operator $\bar{\mathcal{R}}_N$ then becomes

$$\begin{aligned}\bar{\mathcal{R}}_N &= -\sum_{i=1}^{N}\sum_{j=1}^{N} e^{i\kappa_z(h_i-h_N)} \\ &\quad : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_i^t \cdot \bar{\mathbf{I}}_{ij(N)} \cdot \tilde{\mathbf{f}}_{jt} : e^{i\kappa_z(h_j-h_N)}. \end{aligned} \tag{13}$$

The interaction matrix $\bar{\mathbf{I}}_{ij(N)}$ is a function of the number of

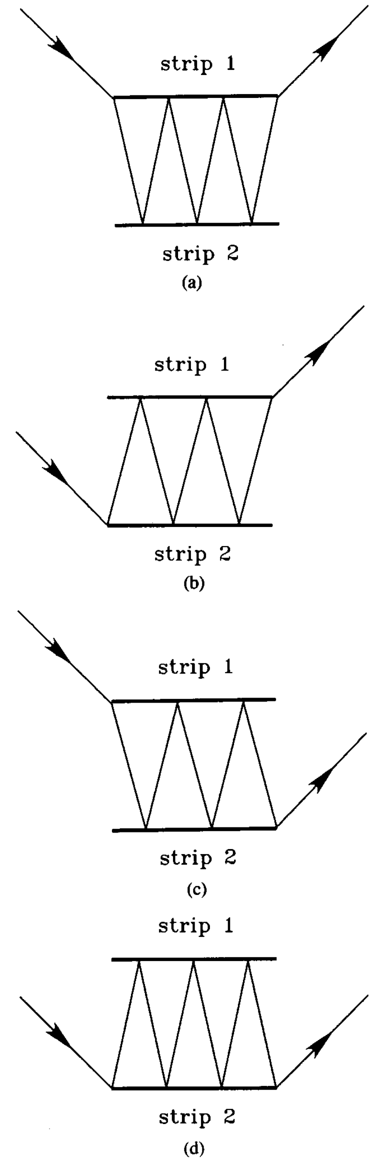


Fig. 2. Scattering interpretation of each term of $\bar{\mathcal{R}}_2$ in (10). (a) First term. (b) Second term. (c) Third term. (d) Last term.

strips N. It is an easily computable quantity because it is discretely indexed. Hence, it is better to derive a recursive relation for $\bar{\mathbf{I}}_{ij(N)}$ since it can be stored in a computer easily. It is fairly easy to obtain a recursive relation among the $\bar{\mathbf{I}}_{ij(k)}$ matrices by making use of the recursive relation for the reflection operators as given by (2). It follows from (13) that the reflection operator $\bar{\mathcal{R}}_{N-1}$ can be written as

$$\begin{aligned}\bar{\mathcal{R}}_{N-1} &= -\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} e^{i\kappa_z(h_i-h_{N-1})} \\ &\quad : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_i^t \cdot \bar{\mathbf{I}}_{ij(N-1)} \cdot \tilde{\mathbf{f}}_{jt} : e^{i\kappa_z(h_j-h_{N-1})}. \end{aligned} \tag{14}$$

Substituting (14) in (3), we find for $\bar{\Gamma}_N$

$$\bar{\Gamma}_N = -\bar{S}_{NN} - \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \bar{S}_{Ni}\cdot\bar{I}_{ij(N-1)}\cdot\bar{S}_{jN}. \quad (15)$$

Next, we substitute (14) in (2) for $\bar{\mathcal{R}}_{N-1}$ and multiply out all the factors to obtain five terms. One of these terms involves fourfold summation since $\bar{\mathcal{R}}_{N-1}$ is multiplied by $\bar{\mathcal{R}}_{N-1}$ in the same term. This term and the first term of (2) can be combined into a single term by manipulating the running variables of the summations with the help of the Kronecker delta function. From the resulting expression, we can extract the $\bar{I}_{ij(N)}$ matrices as

$$\bar{I}_{NN(N)} = \bar{\Gamma}_N^{-1} \quad (16)$$

$$\bar{I}_{iN(N)} = \left(\sum_{m=1}^{N-1}\bar{I}_{im(N-1)}\cdot\bar{S}_{mN}\right)\cdot\bar{\Gamma}_N^{-1}, \quad \text{for } i = 1, 2, \cdots, N-1 \quad (17)$$

$$\bar{I}_{Nj(N)} = \bar{\Gamma}_N^{-1}\cdot\left(\sum_{m=1}^{N-1}\bar{S}_{Nm}\cdot\bar{I}_{mj(N-1)}\right), \quad \text{for } j = 1, 2, \cdots, N-1 \quad (18)$$

$$\bar{I}_{ij(N)} = \bar{I}_{ij(N-1)} + \left(\sum_{m=1}^{N-1}\bar{I}_{im(N-1)}\cdot\bar{S}_{mN}\right)\cdot\bar{\Gamma}_N^{-1}\cdot\left(\sum_{n=1}^{N-1}\bar{S}_{Nn}\cdot\bar{I}_{nj(N-1)}\right), \quad \text{for } i, j = 1, 2, \cdots, N-1. \quad (19)$$

Equations (15)–(19) constitute the recursive relation among the $\bar{I}_{ij(k)}$ matrices. This recursive scheme is actually equivalent to the ordinary moment method where one would have to invert a much larger matrix at once. More precisely, this recursive algorithm is equivalent to first partitioning the larger moment method matrix and then inverting it. Thus, this is an $O(N^3)$ algorithm just like the ordinary moment method is. However, this algorithm has two major advantages over the ordinary moment method. Firstly, noting that each recursion step requires $O(N^2)$ operations, we can efficiently solve the problems where one has to modify some parts of a main body whose solution is already known. If the main body is characterized by n unknowns, and the addition to it by p unknowns, then our recursive method takes $O(n^2p + p^3)$ operations to solve the modified geometry provided that the solution to the main body is known. This problem would take $O[(n+p)^3]$ operations with the ordinary moment method. The same conclusion is reached in [16] using the Sherman–Morrison–Woodbury formula. Another advantage of this algorithm is that it works on smaller partitions of a large matrix, and therefore, is suitable for implementation on computers with virtual memory.

B. Recursive Relation for Fixed Excitation

Once the $\bar{I}_{ij(N)}$ matrices are known, then the scattered field can be calculated. Using (13) in (1), the scattered field is

$$\mathbf{E}_s^R(\mathbf{r}) = -\bar{F}(\mathbf{r}_s) : \sum_{i=1}^{N} e^{i\kappa_z(z+h_i)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_i^t \cdot \underbrace{\sum_{j=1}^{N}\bar{I}_{ij(N)}\cdot\tilde{\mathbf{f}}_{jt} : e^{i\kappa_z(z'+h_j)} : \bar{\mathcal{G}} : \tilde{\mathbf{J}}_s}_{\mathbf{a}_i}. \quad (20)$$

If we let $\tilde{\mathbf{J}}_s = \tilde{\mathbf{f}}_s^t\cdot\mathbf{a}_s$, then

$$\mathbf{a}_i = -\sum_{j=1}^{N}\bar{I}_{ij(N)}\cdot\bar{S}_{js}\cdot\mathbf{a}_s \quad (21)$$

where $\tilde{\mathbf{f}}_i^t\cdot\mathbf{a}_i = \tilde{\mathbf{J}}_i$ is the current amplitude on the ith strip, and $\bar{S}_{js} = -\tilde{\mathbf{f}}_{jt} : e^{i\kappa_z(z'+h_j)} : \bar{\mathcal{G}} : \tilde{\mathbf{f}}_s^t$. We can define

$$\bar{Q}_{is(N)} = \sum_{j=1}^{N}\bar{I}_{ij(N)}\cdot\bar{S}_{js}. \quad (22)$$

Hence for a fixed source $\tilde{\mathbf{J}}_s$ illuminating N strips, it is $\bar{Q}_{is(N)}$ that is needed to calculate the current amplitude on the ith strip. Hence, it is more expedient to rewrite (15)–(19) in terms of $\bar{Q}$ matrices. Consequently,

$$\bar{I}_{NN(N)} = \bar{\Gamma}_N^{-1} = \left(\bar{I} - \sum_{i=1}^{N-1}\bar{Q}_{Ni(1)}\cdot\bar{Q}_{iN(N-1)}\right)^{-1}\cdot\bar{I}_{NN(1)} = \bar{D}^{-1}\cdot\bar{I}_{NN(1)} \quad (23)$$

$$\bar{I}_{iN(N)} = \bar{Q}_{iN(N-1)}\cdot\bar{I}_{NN(N)}, \quad i < N \quad (24)$$

$$\bar{I}_{Nj(N)} = \bar{I}_{NN(N)}\cdot\bar{Q}^T_{Nj(N-1)}, \quad j < N \quad (25)$$

$$\bar{I}_{ij(N)} = \bar{I}_{ij(N-1)} + \bar{Q}_{iN(N-1)}\cdot\bar{I}_{Nj(N)}, \quad i < N, j < N \quad (26)$$

where $\bar{I}_{NN(1)} = -\bar{S}_{NN}^{-1}$, $\bar{Q}_{Ni(1)} = \bar{I}_{NN(1)}\cdot\bar{S}_{Ni}$, and $\bar{Q}^T_{Nj(N-1)} = \sum_{m=1}^{N-1}\bar{S}_{Nm}\cdot\bar{I}_{mj(N-1)}$. By defining $\bar{I}_{ij(N-1)} = 0$ if $j > N-1$, (24) and (26) can be combined to yield

$$\bar{I}_{ij(N)} = \bar{I}_{ij(N-1)} + \bar{Q}_{iN(N-1)}\cdot\bar{I}_{Nj(N)}, \quad i < N, j \le N. \quad (27)$$

Multiplying (27) by $\bar{S}_{js}$ from the right and summing over j from 1 to N, (27) becomes

$$\bar{Q}_{is(N)} = \bar{Q}_{is(N-1)} + \bar{Q}_{iN(N-1)}\cdot\bar{Q}_{Ns(N)}, \quad i < N. \quad (28)$$

We also have

$$\bar{Q}_{Ns(N)} = \sum_{j=1}^{N}\bar{I}_{Nj(N)}\cdot\bar{S}_{js} = \bar{I}_{NN(N)}\cdot\bar{S}_{Ns} + \sum_{j=1}^{N-1}\bar{I}_{Nj(N)}\cdot\bar{S}_{js}, \quad (29)$$

which, from (23) and (25), becomes

$$\bar{Q}_{Ns(N)} = \bar{D}^{-1}\cdot\left(\bar{Q}_{Ns(1)} + \sum_{n=1}^{N-1}\bar{Q}_{Nn(1)}\cdot\bar{Q}_{ns(N-1)}\right) \quad (30)$$

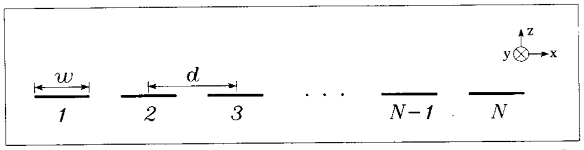


Fig. 3. N parallel, co-planar, identical strips of width w and spacing d.

where $\bar{\mathbf{D}} = \bar{\mathbf{I}} + \sum_{i=1}^{N-1} \bar{\mathbf{Q}}_{Ni(1)} \cdot \bar{\mathbf{Q}}_{iN(N-1)}$. Equations (28) and (30) are two equations for the recursive relations involving $\bar{\mathbf{Q}}$ matrices only. The above algorithm is very similar to the "add-on" technique [14] except that one can add to the problem more than one unknowns at a time using this algorithm. This, too, is an $O(N^3)$ algorithm.

III. Computational Notes

In both of the recursive algorithms of the previous section, the special but important cases of far-field approximation and plane-wave incidence can be handled by letting the observation coordinates and the source coordinates, respectively, go to infinity. Then, stationary phase approximations can be carried out on the first integral (double dot product) and the last integral, respectively, of (1) or (20) to eliminate these integrals.

Although these recursive relations are applicable to a large class of problems, we have programmed the more general arbitrary excitation scheme and mainly used it to treat a special case, namely, the scattering problem from N parallel, coplanar, identical strips of uniform spacing as shown in Fig. 3. This is the case of the finite FSS. Also, when the spacing between the strips is equal to zero, this case serves as an "add-on" example. Sample results due to this problem will be presented in the next section.

There are many symmetry properties that one can exploit in writing an electromagnetic code in order to decrease the computation time. If the basis and testing functions are chosen properly as shown in [2], then we have

$$\bar{\mathbf{S}}_{ij} = \bar{\mathbf{S}}^t_{ji} \tag{31}$$

$$\bar{\mathbf{S}}_{ii} = \bar{\mathbf{S}}^t_{ii} \tag{32}$$

$$\bar{\mathbf{\Gamma}}_i = \bar{\mathbf{\Gamma}}^t_i \tag{33}$$

$$\bar{\mathbf{I}}_{ij(k)} = \bar{\mathbf{I}}^t_{ji(k)}. \tag{34}$$

Thus, not every element of an $\bar{\mathbf{S}}_{ii}$ matrix need be computed due to (32). Also, not every $\bar{\mathbf{S}}_{ij}$ and $\bar{\mathbf{I}}_{ij(k)}$ matrix needs to be computed and stored due to (31) and (34). These symmetry properties are valuable in saving the CPU time. Furthermore, if the strips are identical and uniformly spaced, it can be shown that

$$\begin{matrix} \bar{\mathbf{S}}_{12} & = & \bar{\mathbf{S}}_{23} & = & \bar{\mathbf{S}}_{34} & = \cdots \\ \bar{\mathbf{S}}_{13} & = & \bar{\mathbf{S}}_{24} & = & \bar{\mathbf{S}}_{35} & = \cdots \\ \bar{\mathbf{S}}_{14} & = & \bar{\mathbf{S}}_{25} & = & \bar{\mathbf{S}}_{36} & = \cdots \\ \vdots & & \vdots & & \vdots & \end{matrix} . \tag{35}$$

Hence, if the geometry already contains N strips, with the addition of the $(N+1)$st strip, we have to compute only the $\bar{\mathbf{S}}_{N+1,1}$ matrix but not $\bar{\mathbf{S}}_{N+1,2}, \bar{\mathbf{S}}_{N+1,3}, \cdots, \bar{\mathbf{S}}_{N+1,N}$ matrices. These symmetry properties result in the computation times that are linearly proportional to N, the number of strips in the problem, when N is small. In the small N regime, the matrix filling time dominates over the matrix inversion time. Hence, one would expect to obtain computation times that are proportional to N^2 when N is small. However, illustrated in Fig. 4 are the cases of five basis functions per strip and ten basis functions per strip, both of which consume CPU times that are linear functions of N (for small N) due to the fact that we can easily exploit the above symmetry relations in our algorithm.

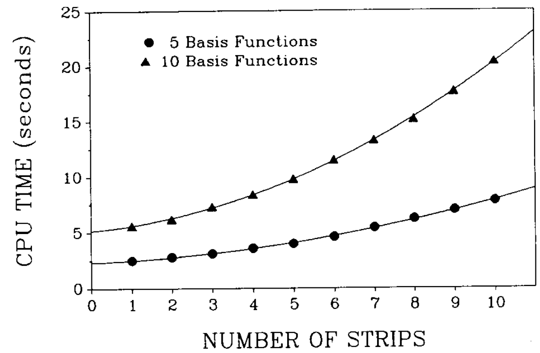


Fig. 4. Computation time as a function of number of strips.

Both of the recursive algorithms presented in the previous section are cousins of the ordinary moment method, and hence, are $O(N^3)$ algorithms. One is tempted to ignore the interactions between distant strips (i.e., assuming interactions only with near neighbors) to speed up these algorithms and obtain $O(N^2)$ algorithms as is the case with the "add-on" technique [15]. However, this should be done with caution. The results in [14] showing current distributions on wide strips have several nonphysical ripples due to the loss of accuracy in this approximation. Our experimentation with such an approximation reveals a similar result.

IV. Results and Conclusion

In this section, in order to illustrate the use of our recursive algorithm, we will present some sample results in the form of radar cross section (RCS) plots and current distribution plots. In the following, the term RCS will be used to mean the relevant bistatic scattering coefficient. One of these scattering coefficients that we will use for both strip and patch cases, namely, σ_{yy}, is defined as

$$\sigma_{yy}(\phi) = k2\pi\rho \left| \frac{E^R_y(\phi, E^I_x = 0)}{E^I_y} \right|^2, \quad \text{for } E^I_y \neq 0 \quad \text{(strip case)} \tag{36a}$$

$$\sigma_{yy}(\theta, \phi) = k^2 4\pi^2 r^2 \left| \frac{E^R_y(\theta, \phi, E^I_x = 0)}{E^I_y} \right|^2, \quad \text{for } E^I_y \neq 0 \quad \text{(patch case)} \tag{36b}$$

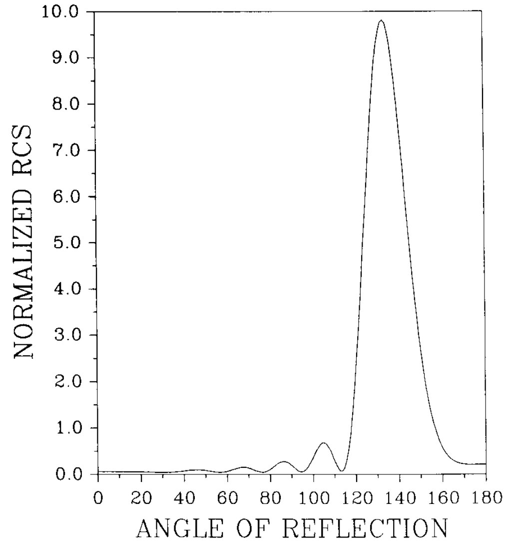


Fig. 5. Indistinguishable plots of RCSs of ten touching strips of $kw = 2$ and a single strip of $kw = 20$.

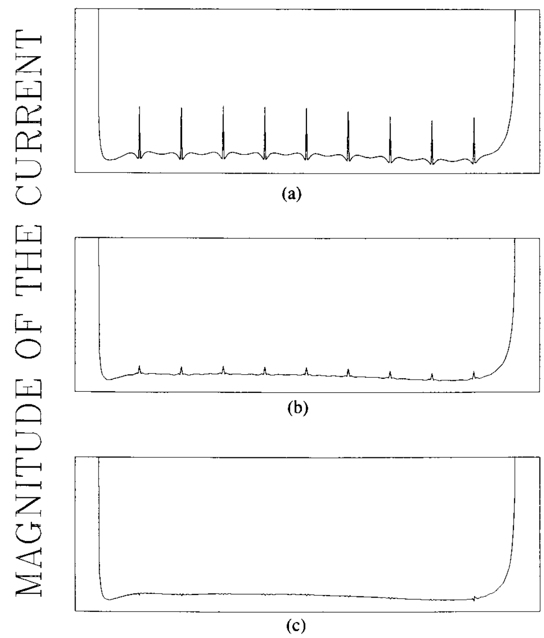


Fig. 6. Convergence of the magnitude of the current distribution on ten touching strips for TM incidence. (a) Five basis functions. (b) Ten basis functions. (c) Fifteen basis functions, per strip.

where E_y^R is the y component of the scattered electric field in the far-field region.

One of the interesting cases that we have worked on is the problem of ten parallel, coplanar strips placed next to each other without gaps or overlaps in between so that these ten strips actually build a much wider single strip. Each strip has a width of $kw = 2$ so that the wider strip has a width of $kw = 20$. A TM (to y) plane wave is incident in the structure at 45°. Fig. 5 shows two indistinguishable plots of normalized RCSs as a function of the angle of reflection using 15 basis functions on the single wide strip and five basis functions per strip on the ten-strip geometry for the longitudinal current. One of these plots corresponds to the RCS due to ten strips of $kw = 2$ each, while the other corresponds to the RCS due to a single strip of $kw = 20$. This example shows that we can use smaller pieces as the building blocks of larger geometries in our recursive algorithm. One should note that the case of touching strips constitutes a more difficult convergence problem than the case in which there are gaps between the strips. This is basically due to the fact that we are using basis functions with edge singularities and these basis functions for the longitudinal current become infinitely large at the strip edges. However, when the strips are touching, the current distribution should be continuous over the complete geometry implying that the singularities of the basis functions should be cancelled on the inner strips. The convergence of the magnitude of the current distribution on ten touching strips to the magnitude of the current distribution on a single wide strip as the number of basis functions gets larger is illustrated in Fig. 6.

An even more severe convergence problem occurs for the case of TE (to y) plane wave incidence. In this case, the current flows in the transverse direction on the strips and the basis functions become zero at the strip edges. When we bring two strips closer together until they start touching, these basis functions introduce a discontinuity in the overall current distribution. One has to use many basis functions before the convergence to the case of a single strip of double width is achieved. For the case of TE plane wave incidence at 45° on two touching strips of $kw = 2$ each, 50 transverse basis functions per strip are needed for convergence. One way to relax this difficult convergence problem is to overlap the two strips. When we overlap the two strips by 1% of the strip width, we have to use 30 basis functions per strip to achieve convergence. In the case of 10% overlap of the strips, 20 basis functions per strip suffice for convergence. The convergence of the magnitude of the transverse current distribution for this case is shown in Fig. 7. The magnitude current distribution given in Fig. 7(d) is exactly that of a single strip of width $kw = 2 \times 2.0 - 0.10 \times 2.0 = 3.8$.

Figs. 8(a) and 8(b) show the normalized RCSs, i.e., $\sigma_{yy}(\phi)$s, when a TM plane wave is incident at 90° and 45°, respectively, on two parallel, coplanar strips of $kw = 2$ each with a separation of $kd = 20$ in between. The nulls and the oscillatory nature of the RCSs are due to the interference pattern created by two radiating strips.

The problem of Fig. 6(c) contains 150 unknowns. However, this is by no means a limit on the number of unknowns that our algorithm can handle. Fig. 9 shows the RCS and the current distribution of a very large geometry composed of 50 touching strips of $kw = 2$ each. We have used nine piecewise linear (triangular or *chapeau*) basis functions on each strip, resulting in a total of 450 unknowns. This, again, is not a limit.

As an illustration of a patch problem, Fig. 10 shows the change of RCS as a co-planar array of four square patches is constructed in steps. As the number of patches grows up,

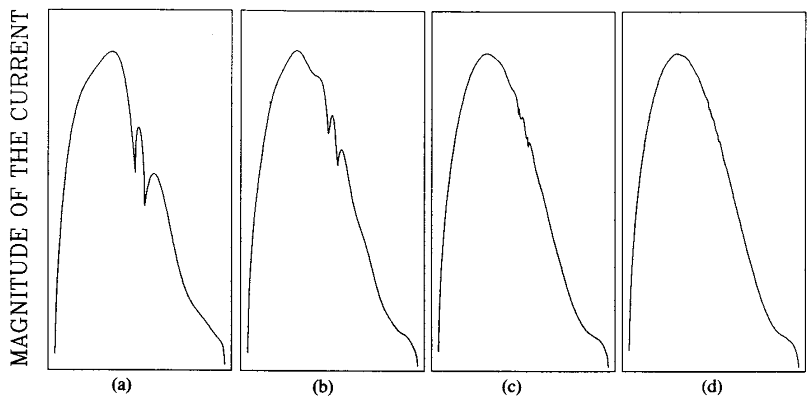


Fig. 7. Convergence of the magnitude of the current distribution on two overlapped strips for TE incidence. (a) Five basis functions, (b) Ten basis functions. (c) Fifteen basis functions, (d) Twenty basis functions, per strip.

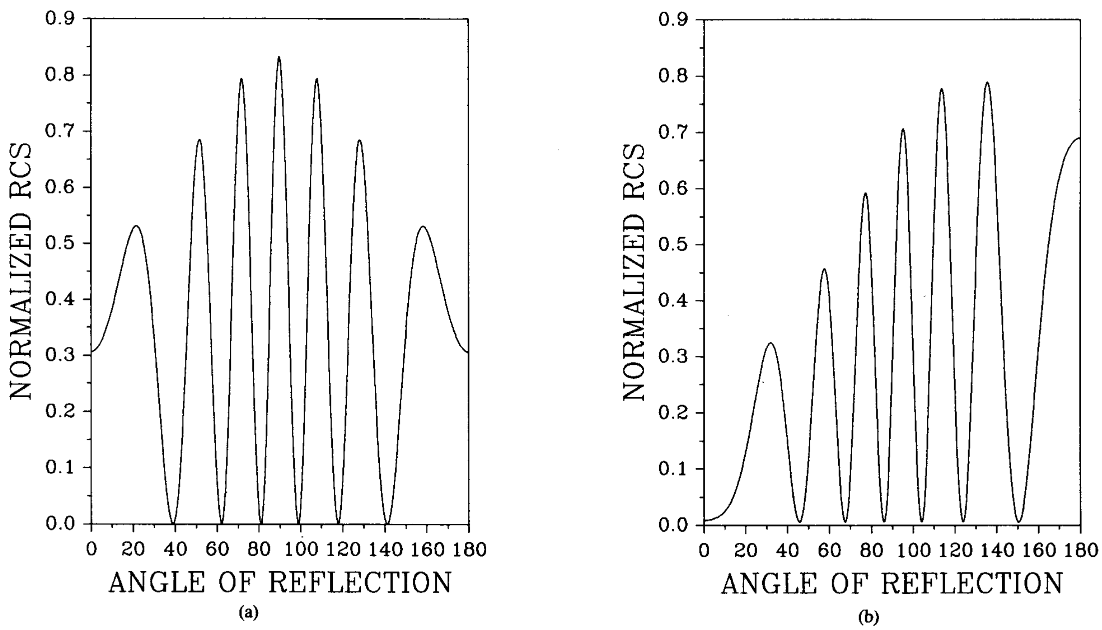


Fig. 8. RCSs of two co-planar strips of $kw = 2$ each with a spacing of $kd = 20$ in between for two different angles of incidence. (a) $\phi_i = 90°$. (b) $\phi_i = 45°$.

the RCS becomes larger and more directive along the primary reflection angle.

Up to this point, we have given sample results of co-planar geometries. As the last example, we would like to present RCS plots of two strips of $kw = 2$ which are placed on top of each other with a spacing of $kd = 4$ as shown in Fig. 11(a). Figs. 11(b) and 11(c) show the plots of $\sigma_{yy}(\phi)$ when a TM plane wave, incident at 90° and 45°, respectively. The reference plane is taken to be the plane of the upper strip. Therefore, these plots present the scattered far field E_y only above this reference plane. Since the geometry is not co-planar, the RCS is not symmetrical above and below this reference plane. However, the RCS below the reference plane can easily be obtained.

In conclusion, we have presented two efficient recursive algorithms to solve the electromagnetic problems involving arbitrary numbers of planar structures in a homogeneous medium. We formulated the scattering problem in this paper, but other problems such as guidance and resonance [2] can be readily formulated once the reflection operator of the geometry is known. This method elucidates the physical picture of the problem. Unlike the conjugate gradient method, each step in the recursion corresponds to a physical geometry. Futher, the physical insight can be used to exploit the various symmetries of the problem.

In this recursion scheme, we need to invert and multiply small matrices (e.g., 15×15) only. Hence, this algorithm can be suited for programming on smaller computers where

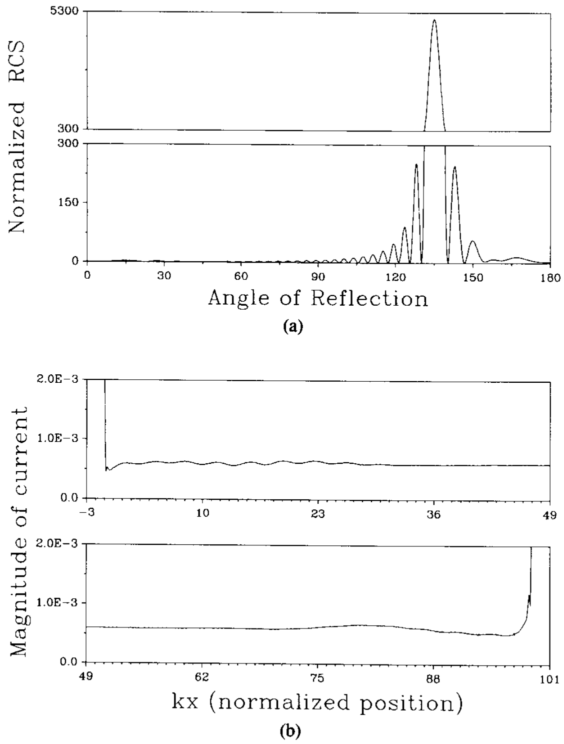


Fig. 9. TM (to y) plane wave is incident at 45° on 50 touching strips of $kw = 2$ each with nine triangular basis functions on each strip. (a) The normalized RCS. (b) Magnitude of the longitudinal current distribution.

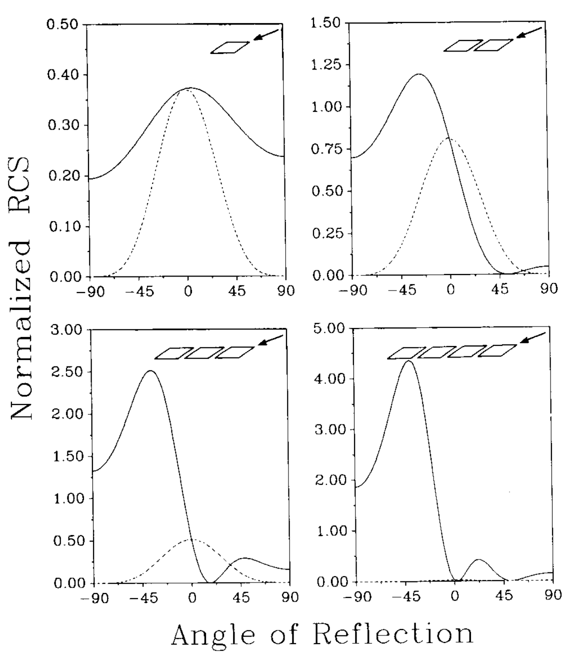


Fig. 10. TM (to y) plane wave is incident at $\theta = 45°$, $\phi = 0°$ on one, two, three, and four co-planar patches. Solid (dashed) line is the RCS on the $\phi = 0°$ ($\phi = 90°$) plane. The patches are square with sides $ka = 2$ and they are separated by $kd = 0.1$.

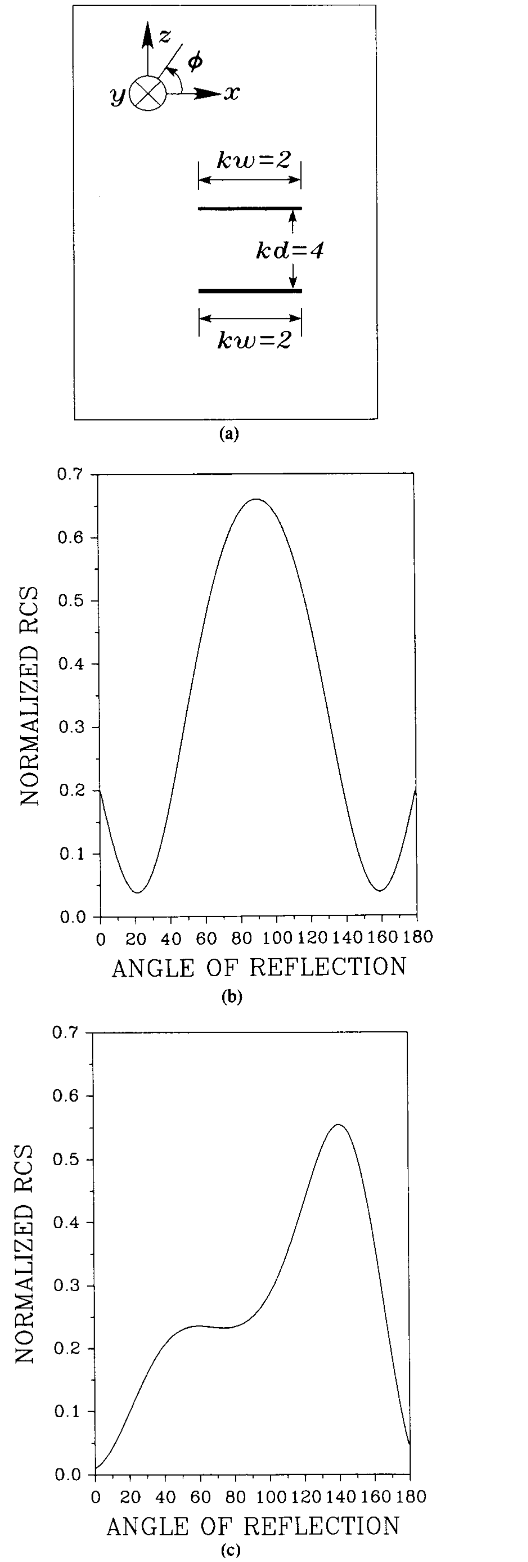


Fig. 11. RCSs of two nonco-planar strips of $kw = 2$ each with a vertical spacing of $kd = 4$ in between for two different angles of incidence. (a) The geometry. (b) $\phi_i = 90°$. (c) $\phi_i = 45°$.

large core memory may not be available. Also, working on one small partition of a large matrix at a time rather than the large matrix itself decreases the number of page faults in a virtual memory machine. This is unlike a direct method of solving the scattering problem by N strips, where the large number of unknowns requires inverting a large matrix at once.

Finally, we note that $\bar{\mathbf{I}}_{ij(k)}$ matrices do not depend on the incident angle. They depend only on the frequency and the dimensions of the geometry. Once these matrices are computed, scattering at various reflection angles due to different incidence angles can easily be obtained. Hence, bistatic RCS computation becomes very efficient.

References

[1] W. C. Chew and L. Gurel, "Reflection and transmission operators for strips or disks embedded in homogeneous and layered media," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 1488-1497, Nov. 1988.

[2] L. Gurel and W. C. Chew, "Guidance and resonance conditions for strips or disks embedded in homogeneous and layered media," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 1498-1506, Nov. 1988.

[3] W. C. Chew and L. Gurel, "Reflection and transmission operators for strips or disks embedded in homogeneous and layered media," Res. Rep. 20, Dec. 1987.

[4] W. C. Chew and B. A. Anderson, "Propagation of electromagnetic waves through geophysical beds in a geophysical environment," *Radio Sci.*, vol. 20, pp. 611-621, May-June 1985.

[5] R. W. Jackson and D. W. Matolak, "Surface-to-surface transition via electromagnetic coupling of coplanar waveguides," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-35, pp. 1027-1032, Nov. 1987.

[6] R. H. Jansen, "Microstrip lines with partially removed ground metallization, theory and applications," *Arch. Elek. Übertragung*, vol. 32, pp. 485-492, Dec. 1978.

[7] T. Itoh, "Spectral domain immittance approach for dispersion characteristics of generalized transmission lines," *IEEE Trans. Microwave Theory Tech.*, vol. MTT-28, pp. 733-736, July 1980.

[8] C. M. Krowne, "Determination of the Green's function in the spectral domain using a matrix method: Application to radiators and resonators immersed in a complex anisotropic medium," *IEEE Trans. Antennas Propagat.*, vol. AP-34, pp. 247-253, Feb. 1986.

[9] W. A. Ko and R. Mittra, "Scattering by a truncated periodic array," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 496-503, Apr. 1988.

[10] T. Cwik and R. Mittra, "The effects of the truncation and curvature of periodic surfaces: a strip grating," *IEEE Trans. Antennas Propagat.*, vol. 36, pp. 612-622, May 1988.

[11] K. Merewether and R. Mittra, "Spectral-domain analysis of a finite frequency-selective surface with cross-shaped conductor patches," in *1988 IEEE Antennas Propagat. Soc. Int. Symp. Dig.*, vol. II, pp. 742-745.

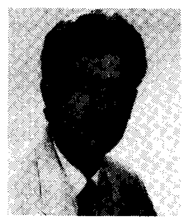
[12] R. Kastner and R. Mittra, "Iterative analysis of finite-sized planar frequency selective surfaces with rectangular patches or perforations," *IEEE Trans. Antennas Propagat.*, vol. AP-35, pp. 372-376, Apr. 1987.

[13] D. M. Pozar, "Finite phased arrays of rectangular microstrip patches," *IEEE Trans. Antennas Propagat.*, vol. AP-34, pp. 658-665, May 1986.

[14] R. Kastner, "An "add-on" method for the analysis of scattering from large planar structures," *IEEE Trans. Antennas Propagat.*, vol. AP-37, pp. 353-361, Mar. 1989.

[15] R. Kastner and R. Mittra, "An "add-on" approach to determining the effect of the modification of the geometry of a thin plate on its electromagnetic scattering problems," in *1988 IEEE Antennas Propagat. Soc. Int. Symp. Dig.*, vol. II, pp. 626-629.

[16] E. Yip and B. Tomas, "Obtaining scattering solutions for perturbed geometries and materials from moment method solutions," *Appl. Comput. Electromagn. Soc. J. and Newsletter*, vol. 3, no. 2, pp. 95-118, Fall 1988.



Levent Gurel (S'87) was born in Izmir, Turkey, on May 12, 1964. He received the B.Sc. degree from Middle East Technical University, Ankara, Turkey, and the M.S. degree from the University of Illinois, Urbana, in 1986 and 1988, respectively, both in electrical engineering.

Currently, he is studying for the Ph.D. degree and is a Research Assistant at the University of Illinois, working on theoretical and computational aspects of electromagnetic problems of inhomogeneous and layered media including planar resonant and guiding structures and frequency selective surfaces.

Weng Cho Chew (S'79-M'80-SM'86), for a photograph and biography please see page 1056 of the August 1988 issue of this Transactions.