ON THE CONNECTION OF T MATRICES AND INTEGRAL EQUATIONS

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I. INTRODUCTION

This paper forms a connection between the two (of the several) major numerical techniques of computational electromagnetics. These are the T-matrix methods (TMM) [1] and the surface- or volume-integral-equation (IE) methods. Both homogeneous (e.g., guidance, resonance) and inhomogeneous (e.g., scattering) equations can be solved with these two methods. In a scattering problem, for example, the immediate product of a TMM is the T matrix of the scatterer, from which the scattered field can be computed, i.e.,

$$\text{TMM} \Longrightarrow \text{T Matrix} \Longrightarrow \text{Scattered Field.} \quad (1)$$

For the same problem, the immediate product of an IE method would be some current distribution (surface or volume, physical or fictitious), from which the scattered field can be computed, i.e.,

$$\text{IE} \Longrightarrow \text{Current Distribution} \Longrightarrow \text{Scattered Field.} \quad (2)$$

In this paper, we will show how to obtain a TMM solution from an IE solution, i.e.,

$$\text{IE} \Longrightarrow \text{Current Distribution} \Longrightarrow \text{T Matrix} \ (\Longrightarrow \text{Scattered Field}), \quad (3)$$

and vice versa, i.e.,

$$\text{TMM} \Longrightarrow \text{T Matrix} \Longrightarrow \text{Current Distribution} \ (\Longrightarrow \text{Scattered Field}). \quad (4)$$

Although both of the methods have long histories, the idea of bridging the two has not attracted much attention since it would be redundant to do (3) or (4) instead of doing either (1) or (2). Since (1), (2), (3), and (4) all give the scattered field as the end-product, is there a reason for prefering (3) or (4) over (1) or (2)?

The answer to the above question is affirmative if one employs a recursive T-matrix algorithm [2], [3] as opposed to a conventional TMM [1]. The recursive T-matrix algorithms can handle multiple scatterers or a single scatterer divided into multiple subscatterers. These algorithms require the knowledge of individual T matrices of subscatters, which can be obtained either by using a conventional TMM that employs extended boundary conditions (EBCs) [1] or by using (3). Once these individual T-matrices are computed, the recursive T-matrix algorithms combine them to compute the T matrix of the whole geometry. Depending on the geometry, this can be achieved in less than $O(N^3)$ operations [3].

The use of (3) is best appreciated when the individual T matrix of a particular subscatterer is difficult, complicated, or simply impossible to compute using a conventional TMM that employs EBCs [1]. One example is the case of an infinitely thin conducting subscatterer, like a conducting strip or patch [3] (see Section II) since the wave functions are singular on the surfaces of the scatterers. On the other hand, (4) is useful when one desires to compute the current distribution at the end of a recursive T-matrix algorithm (see Section III). Due to the lack of space, we will only consider strip problems employing surface IEs in the rest of this paper, however, extensions for other IEs should be obvious.

II. The T Matrix of a Single Strip

We will first formulate the single strip problem using the concepts of surface IEs and method of moments (MOM), and later extract the T matrix for a single scatterer. Recently, van den Berg [4] considered similar ideas, however, we will give a more general formulation.

If $J_{ip}(x_i) = \mathbf{b}_{ip}^t(x_i) \cdot \mathbf{a}_{ip(1)}$ is the unknown current distribution on the ith isolated strip, where $\mathbf{b}_{ip}^t(x_i)$ is a row vector of arbitrary basis functions, and p denotes different polarizations, then the scattered field is given by [3]

$$E_p^S(\rho_i) = \int dx_i'\, G_{pp}(k|\rho_i - \hat{x}x_i'|)\mathbf{b}_{ip}^t(x_i') \cdot \mathbf{a}_{ip(1)} \,. \tag{5}$$

The Green's function, G_{pp}, can be expanded in terms of vectors of cylindrical wave functions, $\boldsymbol{\psi}_p$, and their regular parts, $\Re g\boldsymbol{\psi}_p$, to obtain

$$E_p^S(\rho_i) = \boldsymbol{\psi}_p^t(k\rho_i) \cdot \underbrace{\int dx_i'\, \mathcal{L}_p \Re g \widehat{\boldsymbol{\psi}}_p(kx_i')\mathbf{b}_{ip}^t(x_i')}_{\overline{\mathbf{M}}_{ii}^{pp}} \cdot \mathbf{a}_{ip(1)} \tag{6}$$

where $\mathcal{L}_p$ is a polarization-dependent differential operator [3]. In a T-matrix formalism, the scattered field is given by [2]

$$E_p^S(\rho_i) = \boldsymbol{\psi}_p^t(k\rho_i) \cdot \overline{\mathbf{T}}_{i(1)}^{pp} \cdot \overline{\boldsymbol{\beta}}_{i0}^p \cdot \mathbf{e}_p \tag{7}$$

where $\overline{\mathbf{T}}_{i(1)}^{pp}$ is the T matrix for the ith isolated strip, $\overline{\boldsymbol{\beta}}_{i0}^p$ is a coordinate-translation matrix, and $\mathbf{e}_p$ is a coefficient vector for the incident field. From (6) and (7), we find that

$$\overline{\mathbf{T}}_{i(1)}^{pp} \cdot \overline{\boldsymbol{\beta}}_{i0}^p \cdot \mathbf{e}_p = \overline{\mathbf{M}}_{ii}^{pp} \cdot \mathbf{a}_{ip(1)} \,. \tag{8}$$

$\mathbf{a}_{ip(1)}$ can be solved in terms of $\mathbf{e}_p$ by using a MOM formalism to obtain [3]

$$\mathbf{a}_{ip(1)} = -\underbrace{\left[\int dx_i\, \mathbf{t}_{ip}(x_i) \int dx_i'\, G_{pp}(k|x_i - x_i'|)\mathbf{b}_{ip}^t(x_i')\right]}_{\overline{\mathbf{S}}_{ii}^{pp}}{}^{-1} \cdot \underbrace{\left[\int dx_i\, \mathbf{t}_{ip}(x_i) \Re g \widehat{\boldsymbol{\psi}}_p^t(kx_i)\right]}_{\overline{\mathbf{N}}_{i0}^{pp}} \cdot \mathbf{e}_p \tag{9}$$

where $\mathbf{t}_{ip}(x_i)$ is a vector of arbitrary testing functions. Combining (8) and (9), we have

$$\overline{\mathbf{T}}_{i(1)}^{pp} = -\overline{\mathbf{M}}_{ii}^{pp} \cdot \left(\overline{\mathbf{S}}_{ii}^{pp}\right)^{-1} \cdot \overline{\mathbf{N}}_{ii}^{pp} \,. \tag{10}$$

Once the T matrix for a single strip is computed, one can use it in the recursive algorithms [2], [3] and solve the problem of total scattering from N strips in less than $O(N^3)$ operations.

III. Current Distributions on the Strips

In the presence of N strips, the scattered field is given by

$$E_p^S(\rho) = \sum_{i=1}^{N} \int dx_i'\, G_{pp}(k|\rho - \hat{x}x_i'|)\mathbf{b}_{ip}^t(x_i') \cdot \mathbf{a}_{ip(N)} \tag{11}$$

in an IE formalism, and given by

$$E_p^S(\rho) = \sum_{i=1}^{N} \boldsymbol{\psi}_p^t(k\rho_i) \cdot \overline{\mathbf{T}}_{i(N)}^{pp} \cdot \overline{\boldsymbol{\beta}}_{i0}^p \cdot \mathbf{e}_p \tag{12}$$

in a T-matrix formalism. In order to compute the current distributions on the strips, one needs to find the excitation coefficients, $\mathbf{a}_{ip(N)}$ for all i. Then, $\mathbf{b}^{t}_{ip}(x_i) \cdot \mathbf{a}_{ip(N)}$ gives the current distribution on the ith strip in the presence of $N-1$ other strips. A derivation similar to that given in the previous section results in [3]

$$\mathbf{a}_{ip(N)} = \left(\overline{\mathrm{M}}_{ii}^{pp}\right)^{-1} \cdot \overline{\mathrm{T}}_{i(N)}^{pp} \cdot \overline{\beta}_{i0}^{p} \cdot \mathbf{e}_{p}. \tag{13}$$

The matrix $\overline{\mathrm{M}}_{ii}^{pp}$ is an $M \times N'$ matrix where M is the number of harmonics used for the ith strip and N' is the number of basis functions defined on the ith strip. Although $\overline{\mathrm{M}}_{ii}^{pp}$ is not square, its inverse can still be found via either a least-squares solution or a singular-value-decomposition process to obtain a Moore-Penrose generalized inverse (pseudoinverse) [5].

We have applied the recursive T-matrix algorithms to the electromagnetic problem of scattering from ten strips of width w and spacing $d = 2w$. (see Fig. 1) This is a two-dimensional geometry for a one-dimensional finite-size frequency selective surface. Both TM (to y) and TE (to y) polarized incident plane waves have been considered. Figures 2(a) to 2(e) show the magnitude and the phase of the longitudinal current distributions on the ten strips for the TM case at five different frequencies corresponding to $kw = 1.0, 2.0, \ldots, 5.0$. Similarly, Figs. 3(a) to 3(e) show the transverse current distributions for the TE case at the same frequencies. These results are checked against MOM results with excellent agreements.

References

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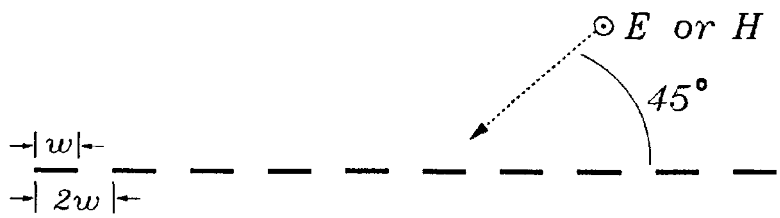

Fig. 1. TM or TE plane wave incident on a finite-size frequency selective surface of ten strips.

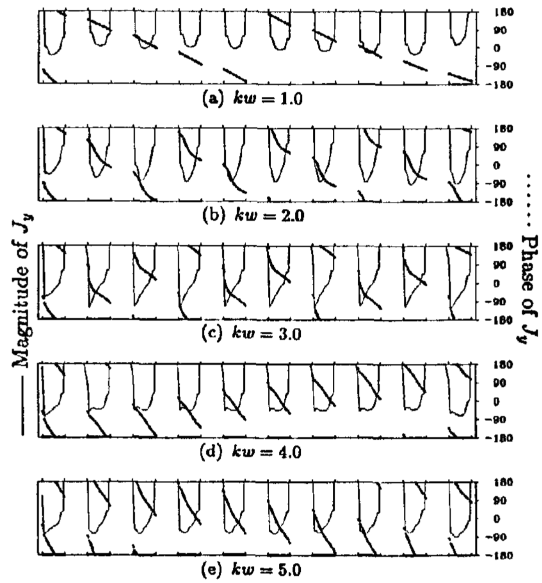

Fig. 2. Magnitude and phase of the longitudinal current (J_y) distribution on the 10-strip FSS at different frequencies: (a) $kw = 1.0$, (b) $kw = 2.0$, (c) $kw = 3.0$, (d) $kw = 4.0$, (e) $kw = 5.0$.

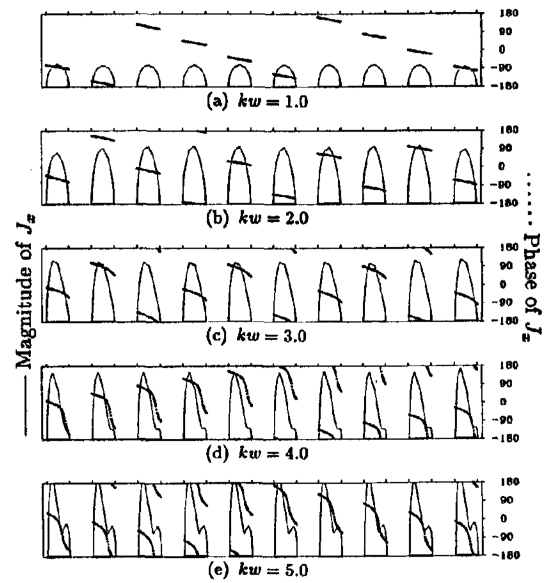

Fig. 3. Magnitude and phase of the transverse current (J_x) distribution on the 10-strip FSS at different frequencies: (a) $kw = 1.0$, (b) $kw = 2.0$, (c) $kw = 3.0$, (d) $kw = 4.0$, (e) $kw = 5.0$.