

Fast Direct (Noniterative) Solvers for Integral-Equation Formulations of Scattering Problems

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1 Introduction

Recently, there has been tremendous progress in the development of fast solvers for the computational electromagnetics. A number of fast iterative solvers [1–7] have been developed as a result of this effort and applied to the solution of large problems using limited computational resources. A drawback of the iterative solvers, which have remarkable performances in solving large problems, is their slow or lack of convergence for the solution of problems that involve resonant or near-resonant structures. For such cases, the number of iterations may become exceedingly large rendering the iterative solver useless. Direct solvers do not share this drawback, however, a straightforward direct solver, such as Gaussian elimination, requires $O(N^3)$ operations to solve an N -unknown problem. This high computational complexity prohibits the solution of large problems with limited resources. Therefore, at least for problems involving resonant or near-resonant structures, what is needed is a direct (noniterative) solver with reduced computational complexity. A family of such solvers will be introduced in this presentation.

In addition to being alternatives to iterative solvers, fast direct solvers can also be used in the framework of iterative solvers as preconditioners and to obtain accurate initial guesses for parts of the geometry. In both cases, the objective is to utilize the direct solvers in a way to accelerate the iterative solvers by reducing the number of iterations required to reach the given convergence criteria. In the case of block-diagonal preconditioners, for instance, the reduced complexity of the fast direct solvers would allow for the solution of blocks with larger sizes, resulting in better preconditioning and thus faster convergence. Similarly, an initial guess that is required to start an iterative scheme can be improved by solving for the critical parts of the geometry using a fast direct solver, and thus reducing the number of iterations.

2 A Family of Fast Direct Solvers

The fast direct solvers that will be presented in this paper are based on the recursive interaction matrix algorithm (RIMA) [8–10] and exploit the aggregation concept of the recursive aggregate T-matrix algorithm (RATMA) [11] to accelerate the solution. Due to the space considerations, only outlines of the algorithms will be given in this summary, saving the details for the presentation.

Consider a matrix equation

$$\bar{\mathbf{S}} \cdot \mathbf{a} = \mathbf{v} \quad (1)$$

obtained by the method-of-moments (MOM) discretization of an integral-equation formulation of a scattering problem. In the above, $\bar{\mathbf{S}}$ is an $N \times N$ matrix, usually referred to as the impedance matrix, \mathbf{a} is a vector of N unknown current coefficients, and \mathbf{v} is an arbitrary right-hand-side (RHS) vector of length N . Equation (1) is also commonly expressed as $\bar{\mathbf{A}} \cdot \mathbf{x} = \mathbf{b}$ and $\bar{\mathbf{Z}} \cdot \mathbf{I} = \mathbf{V}$ using alternative notations, but obviously with the same meaning.

RIMA [8–10] gives an exact solution of Eq. (1) by computing the inverse matrix $\bar{\mathbf{S}}^{-1}$ in $O(N^3)$ operations for any arbitrary geometry. On the other hand, RATMA [11] does not attempt to solve Eq. (1). Instead, RATMA expresses the scattered fields of individual subscatterers in terms of harmonics, aggregates the harmonics, and obtains the overall scattered field of N subscatterers in less than $O(N^3)$ operations. However, some restrictions on the geometry exist. The direct solvers to be presented in this paper combine the advantages of RIMA and RATMA: they have less than $O(N^3)$ computational complexity due to aggregation of the scattered fields, and they are applicable to arbitrary geometries.

2.1 Direct Algorithm to Compute the Scattered Field

This algorithm avoids the solution of Eq. (1) for the unknown vector of current coefficients \mathbf{a} , but instead directly computes the scattered field in $O(NP^2)$ operations. This is accomplished by progressively adding new subscatterers to the problem and exactly computing their interactions with the existing ones, similar to the way RIMA works. However, in order to reduce the computational complexity, the geometry is divided into three regions, whose contents are changing at each recursion step, as depicted in Fig. 1. Assuming that the scattered fields of the subscatterers in the inner zone and the buffer region are already computed, the fields of the subscatterers in the inner zone are aggregated. In the next step, a new subscatterer from the outer zone is added to the problem geometry. The interactions of this new subscatterer with those in the buffer region are computed using RIMA. However, the interactions of the recently added subscatterer with those in the inner zone are computed using a faster method that resembles RATMA. Consequently, a fast technique with no geometry restrictions is obtained for the computation of the scattered field.

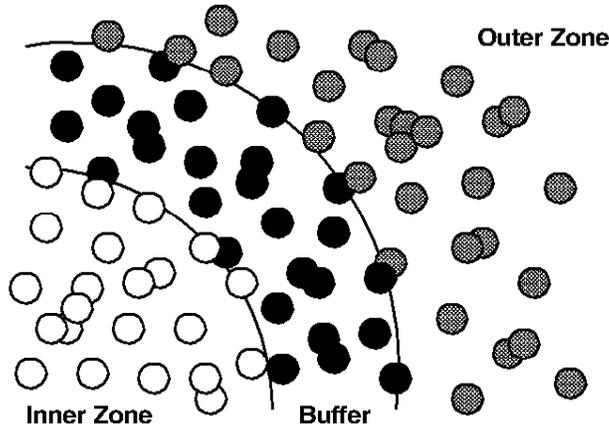


Figure 1: A buffer region is established to separate the inner zone from the outer zone in such a way that the scatterers belonging to inner and outer zones do not overlap at all.

2.2 Direct Algorithm to Invert the Impedance Matrix

It is possible to compute the entries of the inverse of the impedance matrix $\bar{\mathbf{S}}$ in Eq. (1). The computation of the entries of the inverse matrix within a band of width b around the diagonal requires $O(NP^2)$ operations, assuming that the bandwidth b is a constant independent of N . The computation of all the other entries outside of this band requires $O(N^2P)$ operations. Thus, the cost of computing the complete inverse scales as $O(N^2P + NP^2)$. In addition, the multiplication $\mathbf{a} = \bar{\mathbf{S}}^{-1} \cdot \mathbf{v}$ requires $O(N)$ operations if the inverse matrix $\bar{\mathbf{S}}^{-1}$ is banded, and $O(N^2)$ operations if $\bar{\mathbf{S}}^{-1}$ is full, for each distinct RHS vector \mathbf{v} .

The banded portion of the inverse matrix can be utilized in a number of applications. The banded matrix (with the entries outside the band filled with zeros) is an approximation to the full inverse matrix. However, this is an excellent approximation since the nonzero entries of the banded matrix are *exactly* equal to the corresponding entries of the full matrix. This high-quality approximate inverse can be used as a preconditioner in an iterative technique.

2.3 Direct Algorithm to Compute the Current Coefficients

It is also possible to compute directly the unknown vector of current coefficients \mathbf{a} in Eq. (1) without first computing the inverse matrix $\bar{\mathbf{S}}^{-1}$. Once the algorithm outlined in Section 2.1 is completed at a cost of $O(NP^2)$ operations, \mathbf{a} can be computed in $O(NP)$ operations for each distinct \mathbf{v} . Thus, this is a two-step algorithm, where the first step is more expensive but needs to be performed only once, and the second step is less expensive and can be repeated for each distinct RHS vector that depends on the excitation. An analogy can be drawn between this two-step algorithm and the combination of the LU decomposition and the back substitution for the solution of matrix equations.

2.4 Computational Complexities

In the above, computational complexities of the algorithms are expressed in terms of P , which is the number of harmonics required to express the scattered field of a larger scatterer made up of N subscatterers. Clearly then, P depends on N . The exact dependence of P on N is determined by the geometry. Table 1 summarizes this dependence for some interesting cases.

Geometry	Dependence of P	$O(NP^2)$ becomes ...	$O(NP)$ becomes ...
1D	$P \propto N^0$	$O(N)$	$O(N)$
Planar 2D	$P \propto \log N$	$O(N \log^2 N)$	$O(N \log N)$
2D	$P \propto \sqrt{N}$	$O(N^2)$	$O(N^{1.5})$
Planar 3D	$P \propto \sqrt{N} \log N$	$O(N^2 \log^2 N)$	$O(N^{1.5} \log N)$
3D	$P \propto N^{2/3}$	$O(N^{7/3})$	$O(N^{5/3})$

Table 1: Dependence of P on N , and consequently, the ultimate expressions for the computational complexities are determined by the geometry

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