Improving the Accuracy of the Surface Integral Equations for Low-Contrast Dielectric Scatterers†

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Introduction

We consider the solutions of scattering problems involving low-contrast dielectric objects by employing surface integral equations. There are various formulations, which are stable and accurate when the problem involves objects with moderate contrasts with respect to the background medium. However, conventional formulations fail to provide accurate results when the contrast is low. We apply a stabilization procedure based on extracting the non-radiating part of the induced currents so that the remaining radiating currents can be modelled appropriately and the scattered fields from the low-contrast objects can be calculated with improved accuracy. Stabilization is applied to both tangential (T) and normal (N) formulations in order to use the benefits of different formulations. Since the N formulations contain well-tested identity terms, they lead to well-conditioned matrix equations compared to the T formulations. However, these identity terms are responsible for a persistent error, which also remains in the stabilized form of the N formulations. Therefore, stabilized T formulations provide more accurate results, which can be further improved by considering novel techniques to evaluate the right-hand sides (RHS) of the stabilized equations.

Surface Integral Equations

In the construction of the surface integral equations for dielectric objects, the operators for medium $l = 1, 2$ are defined as

$$\mathcal{T}_l\{\boldsymbol{X}\} = ik_l \int_S d\boldsymbol{r'} \left[\boldsymbol{X}(\boldsymbol{r'}) + \frac{1}{k_l^2}\nabla\nabla' \cdot \boldsymbol{X}(\boldsymbol{r'})\right] g_l(\boldsymbol{r}, \boldsymbol{r'}) \tag{1}$$

$$\mathcal{K}_l\{\boldsymbol{X}\} = \int_{PV,S} d\boldsymbol{r'} \boldsymbol{X}(\boldsymbol{r'}) \times \nabla' g_l(\boldsymbol{r}, \boldsymbol{r'}), \tag{2}$$

where $\boldsymbol{X}$ is either electric current ($\boldsymbol{J}$) or magnetic current ($\boldsymbol{M}$) induced on the surface S, k_l is the wavenumber associated with medium l, and $g_l(\boldsymbol{r}, \boldsymbol{r'})$ denotes the homogeneous-space Green's function of the medium l. When $l = 1$ and $l = 2$ correspond to outside and inside of the object, respectively, T formulations can be written as [1]

$$\hat{\boldsymbol{t}} \cdot \left[\begin{array}{c} \left(a\mathcal{T}_1 + b\mathcal{T}_2\right)\{\boldsymbol{J}\} - \left(a\eta_1^{-1}\left[\mathcal{K}_1 - 0.5\mathcal{I}_n\right] + b\eta_2^{-1}\left[\mathcal{K}_2 + 0.5\mathcal{I}_n\right]\right)\{\boldsymbol{M}\} \\ \left(c\eta_1\left[\mathcal{K}_1 - 0.5\mathcal{I}_n\right] + d\eta_2\left[\mathcal{K}_2 + 0.5\mathcal{I}_n\right]\right)\{\boldsymbol{J}\} + \left(c\mathcal{T}_1 + d\mathcal{T}_2\right)\{\boldsymbol{M}\} \end{array}\right] = \hat{\boldsymbol{t}} \cdot \left[\begin{array}{c} a\eta_1^{-1}\boldsymbol{E}^i \\ c\eta_1\boldsymbol{H}^i \end{array}\right], \tag{3}$$

where $\mathcal{I}_n\{\boldsymbol{X}\} = \hat{\boldsymbol{n}} \times \boldsymbol{X}$, $\hat{\boldsymbol{t}}$ is any tangential vector on the surface, and $\hat{\boldsymbol{n}}$ is the outward normal vector. In (3), $\boldsymbol{E}^i$ and $\boldsymbol{H}^i$ are the incident electric and magnetic fields, respectively, and η_l represents the characteristic impedance of the medium $l = 1, 2$. Among different choices for the constants, several combinations give stable formulations, such as

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$\{a = \eta_1, b = \eta_2, c = 1/\eta_1, d = 1/\eta_2\}$ and $\{a = 1, b = 1, c = 1, d = 1\}$, which are known as the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) [2] formulation and the combined T formulation (CTF) [1], respectively. We note that the PMCHWT formulation is free of the identity term while CFT includes weakly-tested identity terms in the case of the Galerkin scheme.

Similar to T formulations, N formulations can be written as

$$\hat{\boldsymbol{n}} \times \begin{bmatrix} \Big(a\big[\mathcal{K}_1 - 0.5\mathcal{I}_n\big] - b\big[\mathcal{K}_2 + 0.5\mathcal{I}_n\big]\Big)\{\boldsymbol{J}\} + \Big(a\eta_1^{-1}\mathcal{T}_1 - b\eta_2^{-1}\mathcal{T}_2\Big)\{\boldsymbol{M}\} \\ -\Big(c\eta_1\mathcal{T}_1 - d\eta_2\mathcal{T}_2\Big)\{\boldsymbol{J}\} + \Big(c\big[\mathcal{K}_1 - 0.5\mathcal{I}_n\big] - d\big[\mathcal{K}_2 + 0.5\mathcal{I}_n\big]\Big)\{\boldsymbol{M}\} \end{bmatrix} = \hat{\boldsymbol{n}} \times \begin{bmatrix} a\boldsymbol{H}^i \\ -c\boldsymbol{E}^i \end{bmatrix}, \tag{4}$$

where different choices for the constants lead to various formulations again. Among these choices, $\{a = 1, b = 1, c = 1, d = 1\}$ results to combined N formulation (CNF) [1], while the combination $\{a = \mu_1, b = \mu_2, c = \epsilon_1, d = \epsilon_2\}$ leads to N-Müller formulation [3]. Using the Galerkin scheme, N formulations include well-tested identity terms, which appear on the diagonal blocks of the matrix equations. Therefore, the conditioning of the N formulations are significantly better than the conditioning of the T formulations [1],[4].

Low-Contrast Breakdown of the Surface Integral Equations

To show the low-contrast breakdown of the surface integral equations, we consider the CTF and CNF formulations. When $\mu_2 \rightarrow \mu_1$ and $\epsilon_2 \rightarrow \epsilon_1$, CTF reduces to

$$\hat{\boldsymbol{t}} \cdot \begin{bmatrix} \eta_1\mathcal{T}_1 & -\mathcal{K}_1 \\ \mathcal{K}_1 & \eta_1^{-1}\mathcal{T}_1 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{M} \end{bmatrix} = 0.5\hat{\boldsymbol{t}} \cdot \begin{bmatrix} \boldsymbol{E}^i \\ \boldsymbol{H}^i \end{bmatrix}, \tag{5}$$

where it can be shown that solution is $\boldsymbol{J} = \hat{\boldsymbol{n}} \times \boldsymbol{H}^i$ and $\boldsymbol{M} = -\hat{\boldsymbol{n}} \times \boldsymbol{E}^i$ [5]. On the other hand, CNF reduces to a more trivial form as

$$\hat{\boldsymbol{n}} \times \begin{bmatrix} -\mathcal{I}_n & 0 \\ 0 & -\mathcal{I}_n \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \mathcal{I} & 0 \\ 0 & \mathcal{I} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{M} \end{bmatrix} = \hat{\boldsymbol{n}} \times \begin{bmatrix} \boldsymbol{H}^i \\ -\boldsymbol{E}^i \end{bmatrix}. \tag{6}$$

We also note that the trivial solution $(\boldsymbol{J}, \boldsymbol{M}) = (\hat{\boldsymbol{n}} \times \boldsymbol{H}^i, -\hat{\boldsymbol{n}} \times \boldsymbol{E}^i)$ is in the null space of the outer problem, i.e.,

$$\begin{bmatrix} \hat{\boldsymbol{t}}\cdot \\ \hat{\boldsymbol{n}}\times \end{bmatrix} \begin{bmatrix} \eta_1\mathcal{T}_1 & -\mathcal{K}_1 + 0.5\mathcal{I}_n \\ \mathcal{K}_1 - 0.5\mathcal{I}_n & \eta_1^{-1}\mathcal{T}_1 \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H}^i \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E}^i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7}$$

In other words, the scattered field reduces to zero as $\mu_2 \rightarrow \mu_1$ and $\epsilon_2 \rightarrow \epsilon_1$. When the contrast is low, both CTF and CNF can formulate the physical problem appropriately. In other words, as the contrast goes to zero, the electric and magnetic surface currents reduce to tangential incident fields, which do not radiate, as indicated in (7). On the other hand, existence of the non-radiating currents causes inaccuracy problems when the formulations are discretized. In general, as the contrast goes down, the total current is composed of a large non-radiating part and a relatively small radiating part, i.e.,

$$\begin{bmatrix} \boldsymbol{J} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H} \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H}^i \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E}^i \end{bmatrix} + \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H}^r \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E}^r \end{bmatrix}. \tag{8}$$

Due to the large non-radiating part $(\boldsymbol{J}^i, \boldsymbol{M}^i) = (\hat{\boldsymbol{n}} \times \boldsymbol{H}^i, -\hat{\boldsymbol{n}} \times \boldsymbol{E}^i)$, the radiating currents $(\boldsymbol{J}^r, \boldsymbol{M}^r) = (\hat{\boldsymbol{n}} \times \boldsymbol{H}^r, -\hat{\boldsymbol{n}} \times \boldsymbol{E}^r)$ cannot be modelled properly even with the sufficient size of the discretization elements with respect to the frequency. Then, the scattered fields contain large errors due to the coarse representation of the radiating part of the total current. Consequently, our stabilization procedure is based on extracting the non-radiating part of the induced currents and calculating the scattered fields directly from the radiating currents [6].

Stabilization of the Surface Integral Equations

For the stabilization of CNF, we extract the non-radiating part of the current and add it to the RHS of the equation to obtain

$$\begin{bmatrix} \mathcal{I} + \hat{\boldsymbol{n}} \times \mathcal{K}_1 - \hat{\boldsymbol{n}} \times \mathcal{K}_2 & \eta_1^{-1} \hat{\boldsymbol{n}} \times \mathcal{T}_1 - \eta_2^{-1} \hat{\boldsymbol{n}} \times \mathcal{T}_2 \\ -\eta_1 \hat{\boldsymbol{n}} \times \mathcal{T}_1 + \eta_2 \hat{\boldsymbol{n}} \times \mathcal{T}_2 & \mathcal{I} + \hat{\boldsymbol{n}} \times \mathcal{K}_1 - \hat{\boldsymbol{n}} \times \mathcal{K}_2 \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H}^r \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E}^r \end{bmatrix} = \\ - \begin{bmatrix} \hat{\boldsymbol{n}} \times \mathcal{K}_1 - \hat{\boldsymbol{n}} \times \mathcal{K}_2 & \eta_1^{-1} \hat{\boldsymbol{n}} \times \mathcal{T}_1 - \eta_2^{-1} \hat{\boldsymbol{n}} \times \mathcal{T}_2 \\ -\eta_1 \hat{\boldsymbol{n}} \times \mathcal{T}_1 + \eta_2 \hat{\boldsymbol{n}} \times \mathcal{T}_2 & \hat{\boldsymbol{n}} \times \mathcal{K}_1 - \hat{\boldsymbol{n}} \times \mathcal{K}_2 \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{H}^i \\ -\boldsymbol{E}^i \end{bmatrix}. \tag{9}$$

In this stable CNF (S-CNF) formulation, radiating currents are solved directly and they can be obtained more accurately. However, since the N formulations contain well-tested identity operators, their results are usually contaminated with a persistent error compared to T formulations [1]. This inaccuracy problem is extensively investigated [7] for the solutions of perfectly conducting objects with the magnetic-field integral equation (MFIE) using the conventional Rao-Wilton-Glisson (RWG) [8] basis functions. Although S-CNF is more stable than CNF, it still includes the identity term. Therefore, to obtain improved accuracy, we also apply a stabilization to the T formulation. In this manner, we consider a balanced PMCHWT (B-PMCHWT) formulation that is completely free of the identity term and also balanced to have identical elements in the diagonal blocks of the matrix equation. Then, we stabilize the formulation as

$$\hat{\boldsymbol{t}} \cdot \begin{bmatrix} (\eta_1 \mathcal{T}_1 + \eta_2 \mathcal{T}_2) & -(\mathcal{K}_1 + \mathcal{K}_2) \\ \eta_1 \eta_2 (\mathcal{K}_1 + \mathcal{K}_2) & (\eta_2 \mathcal{T}_1 + \eta_1 \mathcal{T}_2) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H}^r \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E}^r \end{bmatrix} \\ = \hat{\boldsymbol{t}} \cdot \begin{bmatrix} (\eta_1 \mathcal{T}_1 - \eta_2 \mathcal{T}_2) & -(\mathcal{K}_1 - \mathcal{K}_2) \\ \eta_1 \eta_2 (\mathcal{K}_1 - \mathcal{K}_2) & (\eta_2 \mathcal{T}_1 - \eta_1 \mathcal{T}_2) \end{bmatrix} \cdot \begin{bmatrix} \hat{\boldsymbol{n}} \times \boldsymbol{H}^i \\ -\hat{\boldsymbol{n}} \times \boldsymbol{E}^i \end{bmatrix}, \tag{10}$$

which can be called SB-PMCHWT.

Finally, we note that the stabilized equations in (9) and (10) require the application of the operators on the incident fields. In both S-CNF and SB-PMCHWT, this is achieved by testing the incident fields and expanding them in a series of basis functions using sparse matrices representing the identity operators. However, such a procedure also leads to inaccuracy problems due to the use of the identity terms. Therefore, to further improve the accuracy, we develop a double-stabilized balanced PMCHWT (DSB-PMCHWT), where the coefficients to expand the incident fields are obtained by solving (5). Then, (10) is applied to find the coefficients of the radiating currents.

Results

To demonstrate the improved accuracy provided by the stabilized equations, Fig. 1 presents the results of the scattering problems involving a sphere of radius 30 cm illuminated by a plane wave propagating in free space at 500 MHz. The problems are formulated by various integral equations and discretized with a mesh size of 6 cm, which corresponds to 1/10 of the wavelength outside the sphere. We employ RWG functions and the elements of the matrix equations are obtained with at most 1% error. In Figs. 1(a)-(c), E-plane bistatic radar cross section (RCS) values are depicted for different values of the relative dielectric constant of the sphere, where 0° and 180° correspond to forward-scattered and back-scattered directions, respectively. When the dielectric constant is relatively large, i.e, for $\varepsilon = 1.1$, the values obtained by all formulations are close to the analytical curve obtained by a Mie-series solution. However, as the dielectric constant becomes close to unity and the contrast of the sphere is reduced, we observe that the CNF and B-PMCHWT formulations become inaccurate. We also observe that the stabilized formulations, i.e., S-CNF, SB-PMCHWT, and DSB-PMCHWT, do not breakdown, while DSB-PMCHWT seems to provide the most accurate results. For a clear comparison, Fig. 1(d) presents the relative error in the forward-scattered RCS values as a function of contrast, i.e., $\varepsilon - 1$. it can be observed that S-CNF is more stable compared to CNF, but it contains the inherited error due to the identity term.

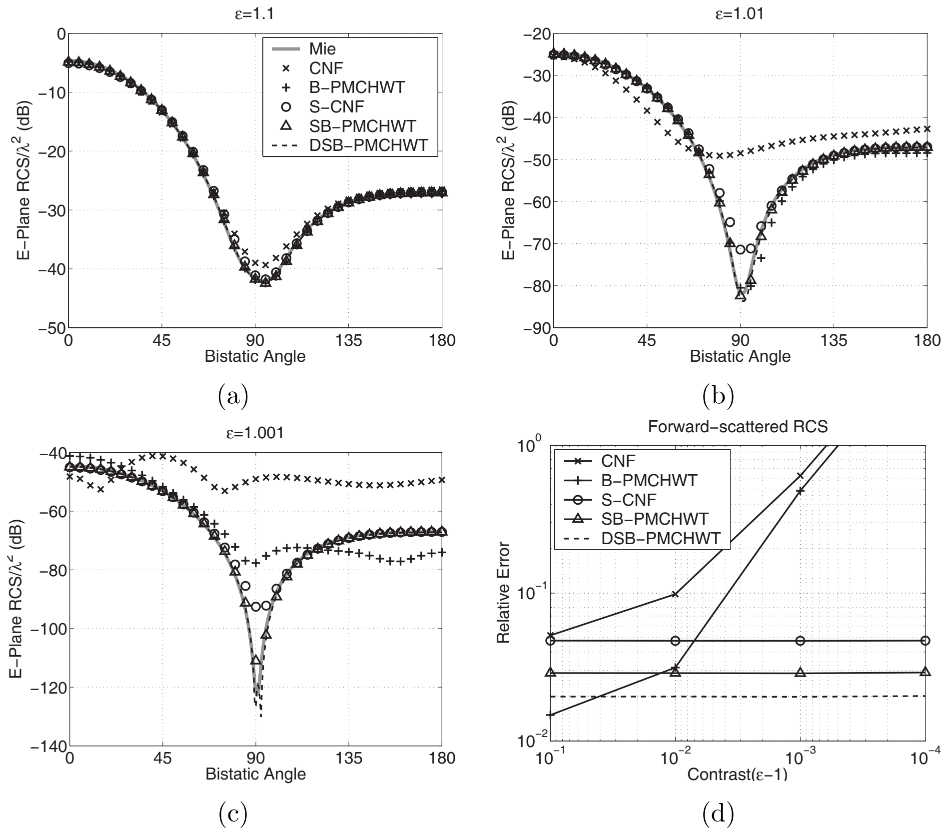

Figure 1: RCS values obtained by various formulations for a sphere of radius 30 cm at 500 MHz when the relative dielectric constant of the sphere is (a) 1.1, (b) 1.01, and (c) 1.001. Relative errors in the forward-scattered RCS values obtained by various formulations (d).

In this manner, SB-PMCHWT is more accurate compared to S-CNF, while it is relatively inaccurate compared to DSB-PMCHWT, which provides the most accurate results since it is both stabilized and completely free of the identity term.

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