Solutions of Large Integral-Equation Problems with Parallel Preconditioned MLFMA*

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1 Introduction

In this paper, we present efficient and accurate solutions of large numerical problems, which are obtained from mathematical formulations of real-life problems of electromagnetics. We consider matrix equations obtained from the discretization of the integral-equation formulations that are solved iteratively by employing parallel multilevel fast multipole algorithm (MLFMA). We mainly focus on the solutions of scattering and radiation problems involving perfectly conducting geometries formulated by electric-field integral equation (EFIE), magnetic-field integral equation (MFIE), and combined-field integral equation (CFIE). The resulting dense matrix equations with large numbers of unknowns are solved on a cluster of relatively inexpensive processors connected via special fast networks. By carefully implementing the parallelization of MLFMA, we obtain high efficiency in the solutions of problems involving various geometries. Enhanced preconditioning schemes are investigated to increase convergence rates, especially for the solutions of ill-conditioned matrix equations that are obtained from EFIE.

2 MLFMA Solutions of Surface Integral Equations

For conducting surfaces, EFIE and MFIE can be written directly from the boundary conditions for the tangential electric and magnetic fields (in phasor notation with the e^{-iwt} convention) as [1]

$$\hat{\boldsymbol{t}} \cdot \int_S d\boldsymbol{r'} \boldsymbol{J}(\boldsymbol{r'}) \cdot \Big(\overline{\boldsymbol{I}} - \frac{\nabla \nabla'}{k^2} \Big) g(\boldsymbol{r}, \boldsymbol{r'}) = \frac{i}{k\eta} \hat{\boldsymbol{t}} \cdot \boldsymbol{E^i}(\boldsymbol{r}) \tag{1}$$

and

$$\boldsymbol{J}(\boldsymbol{r}) - \hat{\boldsymbol{n}} \times \int_S d\boldsymbol{r'} \boldsymbol{J}(\boldsymbol{r'}) \times \nabla' g(\boldsymbol{r}, \boldsymbol{r'}) = \hat{\boldsymbol{n}} \times \boldsymbol{H^i}(\boldsymbol{r}), \tag{2}$$

respectively. In (1) and (2), $\hat{\boldsymbol{t}}$ and $\hat{\boldsymbol{n}}$ are the tangential and outward normal unit vectors on the surface at the observation point $\boldsymbol{r}$, $\boldsymbol{E^i}(\boldsymbol{r})$ and $\boldsymbol{H^i}(\boldsymbol{r})$ are the incident electric and

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magnetic fields, $\boldsymbol{J}$ is the electric current induced on the surface, k is the wavenumber, η is the impedance of free space, and

$$g(\boldsymbol{r},\boldsymbol{r'}) = \frac{e^{ikR}}{4\pi R} \qquad \Big(R = |\boldsymbol{r}-\boldsymbol{r'}|\Big) \tag{3}$$

denotes the free-space Green's function. CFIE is defined as the convex combination of EFIE and MFIE as [2]

$$\text{CFIE} = \alpha\text{EFIE} + (1-\alpha)\frac{i}{k}\text{MFIE}, \tag{4}$$

where α is a parameter between 0 and 1. Choosing a variable α depending on the observation point, we also define hybrid-field integral equation (HFIE) [3], i.e.,

$$\text{HFIE} = \alpha(\boldsymbol{r})\text{EFIE} + \big[1-\alpha(\boldsymbol{r})\big]\frac{i}{k}\text{MFIE}. \tag{5}$$

Different integral-equation formulations are used for open and closed conducting geometries, either by choice or by limitation. MFIE can be formulated only on closed surfaces, whereas EFIE is applicable to both. CFIE is usually preferred over EFIE and MFIE for closed surfaces mainly because it is free of the internal-resonance problem [2] and it generates better-conditioned matrix equations that are crucial for contemporary iterative solvers [4], such as MLFMA [5]. On the other hand, for open surfaces, MFIE and thus CFIE are not applicable, leaving EFIE as the only choice. Unfortunately, EFIE usually produces ill-conditioned matrix equations that are difficult to solve iteratively. For the solutions of composite problems coexisting both open and closed conductors, HFIE provides significant acceleration in the convergence rates of the solutions compared to EFIE [3]. However, EFIE becomes inevitable for the solutions of problems involving only open conductors. Then, we employ strong and efficient preconditioners for the iterative solutions of ill-conditioned matrix equations obtained from EFIE.

By the simultaneous discretization of integral equations and the geometry, we obtain $N \times N$ matrix equations, i.e,

$$\sum_{n=1}^{N} Z_{mn}^{E,M,C,H} a_n = v_m^{E,M,C,H}, \qquad m = 1, ..., N. \tag{6}$$

In the above, a_n represents the unknown coefficients of the basis functions. In addition, $Z_{mn}^{E,M,C,H}$ and $v_m^{E,M,C,H}$ represent the elements of the impedance matrix and the excitation vector, respectively. Matrix equations in (6) are solved iteratively, where the matrix-vector multiplications are accelerated by MLFMA [5] as

$$\overline{\boldsymbol{Z}} \cdot \boldsymbol{x} = \overline{\boldsymbol{Z}}^{NF} \cdot \boldsymbol{x} + \overline{\boldsymbol{Z}}^{FF} \cdot \boldsymbol{x}. \tag{7}$$

In (7), only the near-field interactions denoted by $\overline{\boldsymbol{Z}}^{NF}$ are calculated directly and stored in the memory, while the far-field interactions are computed approximately in a group-by-group manner. Based on the factorization of the Green's functions, aggregation, translation, and disaggregation steps are performed in a multilevel scheme. This way, the overall complexity of the matrix-vector multiplications is reduced to $\mathcal{O}(N \log N)$ for an $N \times N$ dense matrix equation.

3 Efficient Parallelization of MLFMA

Sequential MLFMA employed in an iterative method provides the solution of very large electromagnetic scattering and radiation problems. However, it is also desirable to parallelize MLFMA to further increase the dimensions of the problems solvable on relatively inexpensive platforms, such as clusters of personal computers. On the other hand, communications between the processors and duplications of the computations reduce the efficiency of the parallelization. In this section, we outline our efforts for the efficient parallelization of MLFMA.

3.1 Near-Field and Far-Field Partitioning

In MLFMA, near-field interactions are calculated and stored in the memory before the iterations. Since the number of near-field interactions varies for different rows of the matrix, we employ a load-balance algorithm to distribute the rows among the processors. This way, the interactions are shared equally among the processors and each processor calculates its own interactions independently during the setup. On the other hand, far-field interactions are related to the tree structure and they are distributed among the processors in a different way compared to the near-field partitioning. Consequently, we employ gather-scatter operation in each matrix-vector multiplication to match different partitioning of near-field and far-field interactions.

3.2 Parallelization of Aggregation and Disaggregation

In the aggregation step of the matrix-vector multiplications, the radiated fields of each cluster are aggregated at the centers of the clusters. In the disaggregation step, the incoming fields are evaluated at the centers of the testing clusters. For the low levels of MLFMA, we distribute the clusters among the processors by applying a load-balance algorithm. In general, a level is chosen the share the clusters among the processors. Then, for any cluster in this level, all of its sub-clusters in the lower levels are also assigned to the same processor to avoid communications between the processors. For the upper levels, we switch to another strategy via an all-to-all communication, where the radiated and incoming fields are distributed among the processors [6]. Above the transition level, each cluster is shared by all processors and each processor has the same angular portion of the radiated and incoming fields for all clusters. By distributing the field instead of clusters among the processors, load-balancing is improved in the high levels of the tree structure since the distribution of the samples is easier than the distribution of clusters among the processors for these levels. On the other hand, one-to-one communications are required between the processors in order to perform the interpolations and anterpolations during the aggregation and disaggregation steps.

3.3 Parallelization of Translations

In MLFMA, translations are required to convert the radiated fields calculated during the aggregation process into incoming fields to be used in the disaggregation process. For the higher levels of the tree structure, where the field is distributed among the processors, the translations can be performed without any communication. For the lower levels, some

of the translations are related to the clusters that are assigned to the same processor and they can be completed without any communication. However, there are many other translations that require dense communications since they are related to the interactions of pairs of clusters, which belong to different processors. In general, each processor has radiation data to be sent to all other processors. We carefully handle the data traffic by using a fixed communication map between the processors. For p processors, all of the communications are completed in about p steps, each of which includes simultaneous exchanges of data between pairs of coupled processors. When the processors are coupled, data set of a cluster is transferred from the sender to the receiver if it is required. After a transfer, the receiver processor completes all the translations related to the transferred data. This way, it becomes possible to avoid transferring the same data more than once.

4 Parallel Preconditioning Techniques for Large-Scale Problems

Simple preconditioners that use only diagonals or diagonal blocks of the near-field matrices do not provide sufficient reduction in the iteration counts for the ill-conditioned EFIE matrices. For the CFIE matrices, even though convergence can be achieved with easily parallelized block-diagonal preconditioner (BDP), further reduction in the solution time can be achieved with stronger preconditioners. Omitting some of the entries of the near-field matrix, particularly without considering the numeric values, degrades the performance of the preconditioners significantly. Hence, for large systems, we need preconditioners that use all or at least most of the information provided by the elements of the near-field matrices. However, neither the generation, nor the application of the preconditioner should exceed the $\mathcal{O}(N \log N)$ complexity of MLFMA. In this section, we present various preconditioning schemes to increase convergence rates for EFIE and CFIE solutions of large problems.

4.1 Sparse Approximate Inverse

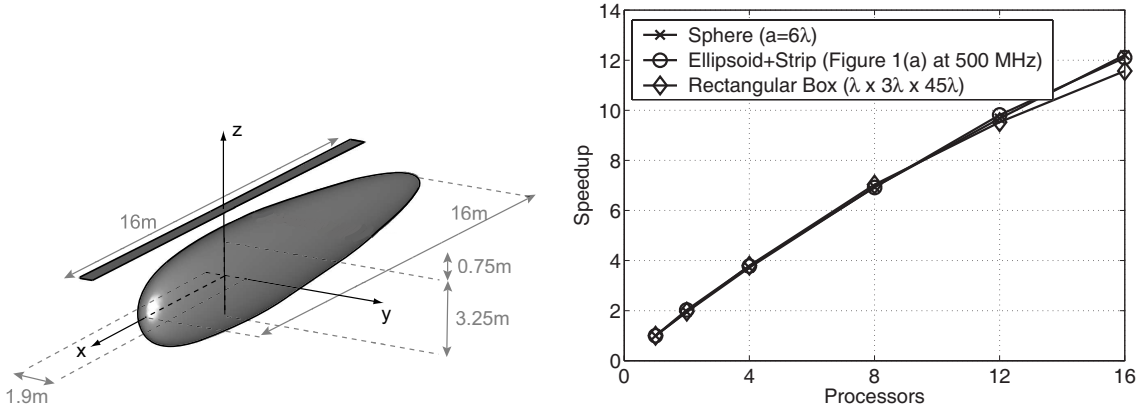
In the sparse appoximate inverse (SAI) preconditioners, the approximation of the inverse of the near-field matrix is performed by minimizing

$$\|\overline{\boldsymbol{I}} - \overline{\boldsymbol{M}} \cdot \overline{\boldsymbol{Z}}^{NF}\|_F. \tag{8}$$

The approximation is achieved by enforcing the preconditioner matrix $\overline{\boldsymbol{M}}$ to be sparse. With the Frobenius norm choice, the minimization can be performed independently for each row by using the identity

$$\|\overline{\boldsymbol{I}} - \overline{\boldsymbol{M}} \cdot \overline{\boldsymbol{Z}}^{NF}\|_F^2 = \sum_{i=1}^{n} \|\boldsymbol{e}_i - \boldsymbol{m}_i \cdot \overline{\boldsymbol{Z}}^{NF}\|_2^2, \tag{9}$$

where $\boldsymbol{e}_i$ is the ith unit row vector and $\boldsymbol{m}_i$ is the ith row of the preconditioner. The nonzero structure of the near-field matrix itself is a natural candidate for the nonzero pattern of SAI. Moreover, using the block structure of the near-field matrix, QR factorization involved in the least squares solutions of (9) can be done performed for each diagonal block [7]. In this way, the construction time of the preconditioner can be reduced substantially.



(a) (b)

Figure 1: (a) A geometry including a long strip over an ellipsoid, which is used in the tests of the parallelization of MLFMA. (b) Speedup obtained by the parallelization of MLFMA for the solutions of three different scattering problems, i.e., a sphere of radius 6λ, the geometry described in Figure 1(a) at 500 MHz, and a long rectangular box with the dimensions of $\lambda \times 3\lambda \times 45\lambda$.

4.2 Nested Preconditioning Based on SAI

SAI preconditioners usually do not provide close approximations to the near-field matrices for EFIE. Hence, we propose to use SAI for the iterative solution of the near-field system and then the near-field solution is used as a preconditioner to the original system. For this purpose, a flexible solver is used for the original system, which allows preconditioning operation to change from iteration to iteration. This nested-solvers scheme yields better preconditioners compared to using SAI alone.

4.3 Preconditioning Based on Incomplete MLFMA

For very large problems, the near-field matrices become increasingly sparser and we need further information than that provided by the near-field matrix. Since the preconditioning operation can be performed iteratively with flexible solvers, MLFMA can also be considered for the preconditioning. We perform the matrix-vector multiplications faster by lowering the sampling densities of MLFMA. This way, the cost of the this effective preconditioning operation is significantly reduced.

5 Results

In this section, we present examples for the solutions of large problems, which are obtained by employing parallel MLFMA implementations on an 8-way SMP server with dual-core AMD Opteron processors. To show the efficiency of the parallelization of MLFMA, Figure 1 compares the speedups for the scattering problems related to a sphere of radius 6λ, the geometry depicted in Figure 1(a) at 500 MHz, and a rectangular box with dimensions of $\lambda \times 3\lambda \times 45\lambda$. In all cases, the parallelization provides about 12 speedup for 16 processors. In other words, independent of the geometry, 75% efficiency is achieved in the parallel solutions.

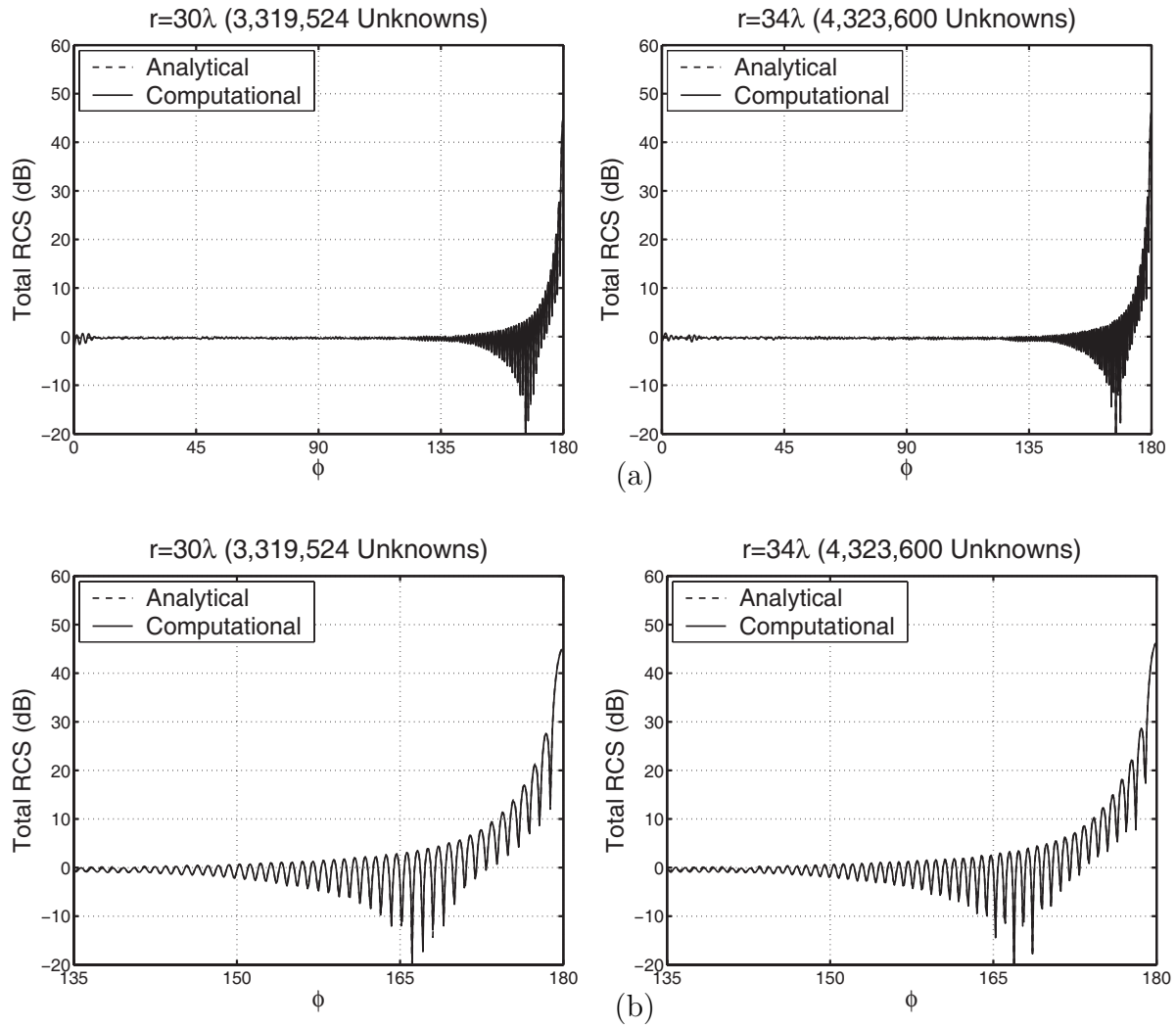


Figure 2: Bistatic RCS in dB of spheres of radius 30λ and 34λ calculated on the E-plane with respect to bistatic direction (a) from 0^o (backscattering) to 180^o (forward-scattering) and (b) from 135^o to 180^o. Analytical Mie-series results are also plotted to check the accuracy of the computations.

As an example to the solutions of large problems, Figure 2 presents the results of scattering problems related to spheres of radius 30λ and 34λ. The spheres are illuminated by a plane wave and Figure 2(a) depicts the bistatic radar cross section (RCS) in decibels (dB) on the E-plane. For a clear comparison, the same data is plotted from 135^o to 180^o in Figure 2(b). All figures also include the analytical values obtained by a Mie series solution to show that the computational results are accurate. Using CFIE with BDP, residual error drops under 10^{-6} in only 29 and 33 iterations for the spheres of radius 30λ and 34λ, respectively. For the smaller problem with 3,319,524 unknowns, the solution requires maximum 1750 MBytes and completed in about 7500 seconds. For the larger problem with 4,323,600 unknowns, 2700 MBytes of memory is required while the solution is achieved in 11,050 seconds.

As effective preconditioners, we compare BDP, SAI, and the iterative preconditioner obtained with an incomplete MLFMA, in which the inner system solution is also preconditioned with SAI. We denote this preconditioner with IFMM/SAI. For EFIE, we compare SAI, a preconditioner obtained from the iterative solution of the near-field system, which we name NF/SAI, and IFMM/SAI. GMRES and FGMRES are used as the solvers and the convergence criterion is determined as a six order decay in the initial residual. The criterion for the inner system solution of IFMM/SAI is fixed as only a one order of decay in the initial residual or a maximum of 10 iterations. For NF/SAI, the stopping criteria for inner system solution is fixed as one order of decay in the initial residual or a maximum of 5 iterations. The experimental results reveal that such loose convergence criteria produce very effective preconditioners.

In Figure 3, we show the solution times of the CFIE matrices for large problems, where the numbers of unknowns exceed 100,000, and for very large problems, where the unknowns exceed 700,000. For both problem classes, SAI reduces the solution time by about $20\% - 30\%$, and IFMM/SAI reduces the solution time by a factor of 3, compared to the commonly used BDP. Using effective preconditioning strategies such as IFMM/SAI, we achieve the solution of a complex geometry such as a stealth target named Flamme [8] with approximately 1,000,000 unknowns in about 40 minutes. Moreover, IFMM/SAI have negligible memory requirement for storing the preconditioned Krylov subspace vectors [9].

In Figure 4, we present the results related to highly insolvable EFIE matrices. We note that convergence cannot be attained in these problems using simple preconditioners such as BDP. SAI causes all iterations to converge in reasonable times. When we use the information carried by the near-field matrices more efficiently with NF/SAI, we reduce the solution times significantly, and when we use the incomplete MLFMA as a preconditioner, the solutions are achieved approximately two times faster compared to SAI.

6 Conclusion

Accurate and efficient solutions of large problems in computational electromagnetics require extensive investigations of various components, such as integral-equation formulations, choice of basis and testing functions, iterative solvers, preconditioners, parallel architectures, and parallelization strategies. In this work, we present our efforts to develop improved strategies in order to solve various problems involving millions of unknowns.

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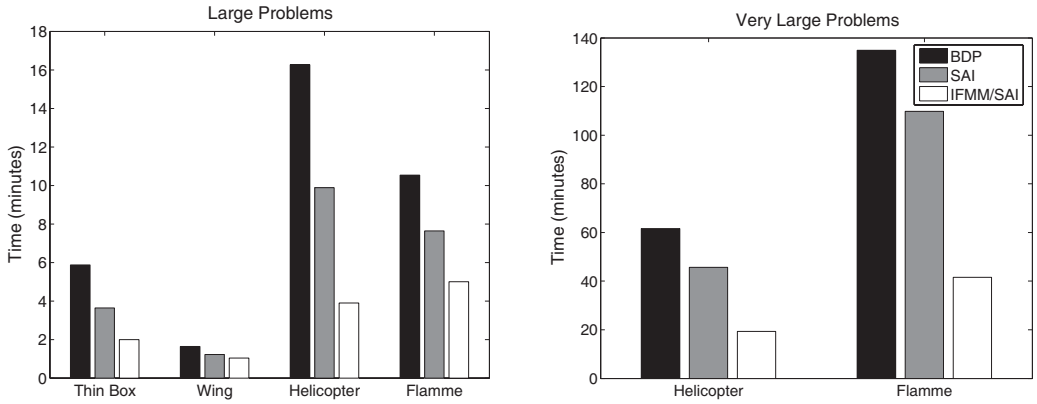

Figure 3: CFIE results for "large problems" and for "very large problems." In the "large problem" category, the thin box has 147,800 unknowns, the wing has 117,945 unknowns, the helicopter has 183,546 unknowns, and the Flamme has 197,892 unknowns. In the "very large problem" category, the helicopter has 739,404 unknowns and the Flamme has 895,407 unknowns.

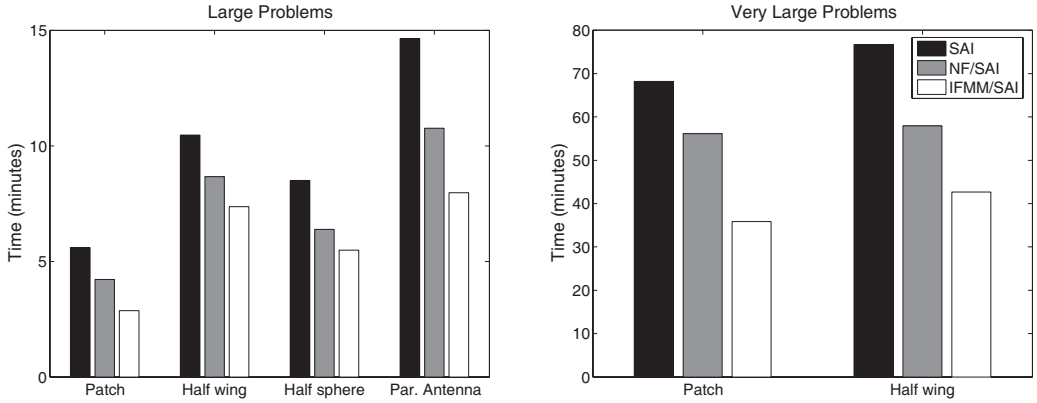

Figure 4: EFIE results for "large problems" and for "very large problems." In the "large problem" category, the patch has 137,792 unknowns, the half wing has 127,925 unknowns, the half sphere has 116,596 unknowns, and the parabolic antenna has 356,439 unknowns. In the "very large problem" category, the patch has 719,000 unknowns and the half wing has 409,514 unknowns.

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